On the Continuous Dependence of Fixed Points on Parameters*

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1 Introduction

In dynamic programming the value function is the fixed point of a Bellman operator T_α on a function space, where α collects preferences, technology and the remaining primitives (Stokey et al., 1989). Comparative statics, continuity of the policy correspondence and consistency of structural estimators all presuppose that

$$\alpha \longmapsto z(\alpha) = \text{the fixed point of } T_\alpha$$

is continuous. Theorem 7.18 of de la Fuente (2000) asserts this under hypotheses that do not suffice in infinite dimensions —precisely the setting of dynamic programming.

Section 2 locates the gap, following Shannon (2026): the printed proof uses a modulus common to all parameters, which the hypotheses do not supply. Proposition 3.1 then constructs a family in ℓ^2 meeting every hypothesis whose solution function jumps, so the statement is false and not merely unproved. Two conditions restore it. Theorem 4.1 replaces the uniform bound by one holding near the parameter of interest, $\limsup_{\alpha\to\alpha_0} \beta_\alpha < 1$; this is what the comparative-statics exercise in the discount factor requires and the uniform bound excludes. Theorem 5.3 shows that when closed balls are compact —in particular when $\dim X < \infty$— no hypothesis on the moduli is needed.

Notation. (X, d) and (Ω, ρ) are metric spaces, $T : X \times \Omega \to X$, $T_\alpha = T(\cdot, \alpha)$, and $\beta_\alpha = \operatorname{Lip}(T_\alpha) = \sup_{x\neq y} d(T_\alpha x, T_\alpha y)/d(x, y)$. We write $z(\alpha)$ for the fixed point of T_α when it exists and is unique, $B[x; r] = \{y \in X : d(y, x) \leq r\}$, and $\|\cdot\|_\infty$ for the supremum metric. That T is continuous in α means that $\alpha \mapsto T(x, \alpha)$ is continuous for each fixed $x \in X$.

2 The statement

Theorem 2.1 (7.18 in de la Fuente, 2000, pp. 88–89)**.** *Let (X, d) and (Ω, ρ) be metric spaces and $T : X \times \Omega \to X$. If (X, d) is complete, T is continuous in α, and for each $\alpha \in \Omega$ the map $T_\alpha = T(\cdot, \alpha)$ is a contraction, then the solution function $z : \Omega \to X$ assigning to each α the fixed point of T_α, $x^* = z(\alpha)$, is continuous.*[1]

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[1]The statement is de la Fuente's, renotated: the source prints "f" where T is meant, and attaches no definition of continuity in α. The latter is immaterial here, since the counterexample of Section 3 satisfies both the pointwise and the joint reading.

Theorem 2.2 (7.18′, uniform contraction principle)**.** *Same hypotheses, with in addition a single $\beta \in (0,1)$ such that $\beta_\alpha \leq \beta$ for every $\alpha \in \Omega$. Then z is continuous.*

Theorem 2.2 is the uniform contraction principle. That it, and not Theorem 2.1, is what de la Fuente's proof establishes was observed by Shannon (2026): the printed hypotheses allow β_α to vary with α, so $\sup_{\alpha\in\Omega} \beta_\alpha = 1$ is not excluded. Taking $\alpha_n \to \alpha$ and inserting $T_{\alpha_n} z(\alpha)$ in the triangle inequality,

$$d\left[z(\alpha_n), z(\alpha)\right] = d\left[T_{\alpha_n} z(\alpha_n), T_\alpha z(\alpha)\right] \leq \beta_{\alpha_n} \, d\left[z(\alpha_n), z(\alpha)\right] + d\left[T_{\alpha_n} z(\alpha), T_\alpha z(\alpha)\right],$$

and absorbing the first term on the right,

$$d\left[z(\alpha_n), z(\alpha)\right] \;\leq\; \frac{1}{1-\beta_{\alpha_n}} \, d\left[T_{\alpha_n} z(\alpha), T_\alpha z(\alpha)\right]. \tag{1}$$

The absorption is the gap. The proof writes a fixed β where β_{α_n} stands, but the hypotheses give only $\beta_{\alpha_n} < 1$ for each n, and the factors $1/(1-\beta_{\alpha_n})$ need not stay bounded; Theorem 2.2 assumes exactly what is used. What breaks down is the argument, not yet the conclusion: Section 3 shows the conclusion fails as well, and Sections 4 and 5 ask how much of the uniform bound can be given up.

3 A counterexample in infinite dimensions

Let $X = \ell^2$ with orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, and let $\Omega = \{0\} \cup \{1/n : n \geq 2\} \subset \mathbb{R}$ carry the usual metric. Both are complete metric spaces (Brezis, 2011).

Fix integers $m_n \to \infty$ with disjoint blocks, say $m_n = 2^n$, and for each $n \geq 2$ define the linear operator A_n by

$$A_n e_i = \begin{cases} e_{i+1}, & m_n \leq i \leq m_n + n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Set $\beta_n = 1 - \frac{1}{n}$ and

$$c_n = \left(\sum_{k=0}^{n} \beta_n^{2k} \right)^{-1/2}. \tag{2}$$

The scalars c_n tend to zero. Summing the geometric series,

$$\sum_{k=0}^{n} \beta_n^{2k} = \frac{1-\beta_n^{2(n+1)}}{1-\beta_n^2}, \qquad 1-\beta_n^2 = \frac{2n-1}{n^2}, \qquad \beta_n^{2(n+1)} = \left(1-\tfrac{1}{n}\right)^{2(n+1)} \longrightarrow e^{-2},$$

so that $\sum_{k=0}^{n} \beta_n^{2k} \sim \frac{1-e^{-2}}{2}\, n$ and therefore

$$c_n \;\sim\; \left(\tfrac{2}{1-e^{-2}}\right)^{1/2} n^{-1/2} \;\longrightarrow\; 0. \tag{3}$$

The family is

$$T\left(x, \tfrac{1}{n}\right) = \beta_n A_n x + c_n e_{m_n}, \qquad T(x,0) = 0. \tag{4}$$

Proposition 3.1. *Family* (4) *satisfies every hypothesis of Theorem 2.1, yet* $\|z(1/n)\| = 1$ *for all* n *while* $z(0) = 0$*. Consequently* z *is discontinuous at* $\alpha = 0$ *and Theorem 2.1 is false.*

Proof. **Step 1:** $\|A_n\| = 1$. For $x \in \ell^2$ we have $A_n x = \sum_{i=m_n}^{m_n+n-1} x_i\, e_{i+1}$, so by orthonormality

$$\|A_n x\|^2 = \sum_{i=m_n}^{m_n+n-1} |x_i|^2 \;\le\; \|x\|^2,$$

with equality at $x = e_{m_n}$. Hence $\|A_n\| = \sup_{\|x\|\le 1} \|A_n x\| = 1$.

Step 2: A_n **is nilpotent,** $A_n^{n+1} = 0$**.** By induction, $A_n^k e_{m_n} = e_{m_n+k}$ for $0 \le k \le n$. Since A_n annihilates every e_i with $i > m_n + n - 1$, in particular $A_n e_{m_n+n} = 0$, whence $A_n^{n+1} e_{m_n} = 0$. On every other basis vector the operator vanishes within at most n steps, so $A_n^{n+1} = 0$.

Step 3: each T_α **is a contraction.** By Step 1,

$$\left\|T\left(x, \tfrac{1}{n}\right) - T\left(y, \tfrac{1}{n}\right)\right\| = \beta_n \|A_n(x-y)\| \le \beta_n \|x - y\|, \qquad \beta_n = 1 - \tfrac{1}{n} < 1,$$

and $T_0 \equiv 0$ is a contraction of modulus 0. Note that $\sup_n \beta_n = 1$.

Step 4: T **is continuous in** α**.** The points $1/n$ are isolated in Ω, so only $\alpha = 0$ requires verification. Fix $x \in \ell^2$. By Step 1,

$$\|A_n x\|^2 = \sum_{i=m_n}^{m_n+n-1} |x_i|^2 \;\le\; \sum_{i \ge m_n} |x_i|^2 \longrightarrow 0,$$

since this is a tail of the convergent series $\|x\|^2$ and $m_n \to \infty$. Together with $c_n \to 0$ from (3),

$$\left\|T\left(x, \tfrac{1}{n}\right) - T(x, 0)\right\| \le \|A_n x\| + c_n \longrightarrow 0.$$

The map T is in fact jointly continuous. Every T_α is 1-Lipschitz in x by Step 3, so for $y \in \ell^2$,

$$\left\|T\left(y, \tfrac{1}{n}\right) - T(x,0)\right\| \;\le\; \left\|T\left(y, \tfrac{1}{n}\right) - T\left(x, \tfrac{1}{n}\right)\right\| + \left\|T\left(x, \tfrac{1}{n}\right) - T(x,0)\right\| \;\le\; \|y - x\| + \|A_n x\| + c_n,$$

which tends to 0 as $(y, 1/n) \to (x, 0)$. The family therefore refutes Theorem 2.1 under either reading of its continuity hypothesis, the pointwise one or the joint one.

Step 5: computation of the fixed point. The fixed-point equation $x = \beta_n A_n x + c_n e_{m_n}$ reads $(I - \beta_n A_n)x = c_n e_{m_n}$. Since $A_n^{n+1} = 0$ by Step 2, the finite sum $S_n = \sum_{k=0}^n \beta_n^k A_n^k$ satisfies, by telescoping,

$$(I - \beta_n A_n)\, S_n \;=\; S_n\, (I - \beta_n A_n) \;=\; I - \beta_n^{n+1} A_n^{n+1} \;=\; I,$$

which exhibits the inverse explicitly; no spectral radius has to be estimated and no series has to be summed. Therefore

$$z(1/n) = (I - \beta_n A_n)^{-1} (c_n e_{m_n}) = c_n \sum_{k=0}^{n} \beta_n^k e_{m_n+k}.$$

Using $A_n e_{m_n+k} = e_{m_n+k+1}$ for $k \le n-1$ and $A_n e_{m_n+n} = 0$,

$$T\left(z(1/n), \tfrac{1}{n}\right) = c_n \sum_{k=0}^{n-1} \beta_n^{k+1} e_{m_n+k+1} + c_n e_{m_n} = c_n \sum_{j=0}^{n} \beta_n^j e_{m_n+j} = z(1/n).$$

Step 6: $\|z(1/n)\| = 1$**.** The vectors $e_{m_n}, \ldots, e_{m_n+n}$ are orthonormal, so the inner product of $z(1/n)$ with itself has no cross terms:

$$\|z(1/n)\|^2 = \langle z(1/n), z(1/n)\rangle = c_n^2 \sum_{k=0}^{n} \beta_n^{2k} = 1$$

by the definition (2) of c_n. Since $z(0) = 0$, we get $\|z(1/n) - z(0)\| = 1$ for every n, while $1/n \to 0$ in Ω. $\square$

The fixed points have disjoint supports, so $\langle z(1/n), z(1/m)\rangle = 0$ and $\|z(1/n) - z(1/m)\| = \sqrt{2}$ for $n \neq m$: the sequence $\{z(1/n)\}$ is bounded and has no convergent subsequence.

4 The modulus near the limiting parameter

The uniform bound of Theorem 2.2 enters the proof only through the denominator of (1), and continuity is a local property. It is therefore enough that the bound hold near the parameter at which continuity is asserted.

Theorem 4.1 (local version)**.** *Let (X, d) be complete, let T be continuous in α, and let T_α be a contraction with modulus $\beta_\alpha < 1$ for every $\alpha \in \Omega$. If*

$$\limsup_{\alpha\to\alpha_0} \beta_\alpha < 1,$$

then z is continuous at α_0.

Proof. If α_0 is isolated in Ω, some ball around it meets Ω only at α_0, every map into X is continuous there, and there is nothing to prove; the punctured suprema below are then over empty sets, so the hypothesis holds vacuously under the convention $\sup\emptyset = -\infty$. Assume therefore that α_0 is not isolated, so that those sets are nonempty and the suprema are real numbers in $[0, 1]$.

Write

$$L = \limsup_{\alpha\to\alpha_0} \beta_\alpha = \inf_{\delta>0}\, \sup\left\{\beta_\alpha :\ 0 < \rho(\alpha, \alpha_0) < \delta\right\},$$

which is < 1 by hypothesis, and put $\lambda = \frac{1}{2}(L + 1)$, so that $L < \lambda < 1$. Since L is the infimum over $\delta > 0$ of the quantities on the right and $L < \lambda$, some $\delta > 0$ satisfies

$$\sup\left\{\beta_\alpha :\ 0 < \rho(\alpha, \alpha_0) < \delta\right\} \ \leq\ \lambda.$$

Let $U = \{\alpha \in \Omega : \rho(\alpha, \alpha_0) < \delta\}$. The displayed bound covers every point of U except α_0 itself, whose modulus is $\beta_{\alpha_0} < 1$ by hypothesis, so

$$\beta = \max\left\{\lambda,\ \beta_{\alpha_0}\right\}$$

is a single $\beta < 1$, being the maximum of two numbers each < 1, with $\beta_\alpha \leq \beta$ for every $\alpha \in U$.

Now $T|_{X\times U}$ satisfies the hypotheses of Theorem 2.2 with this β, and the fixed point of T_α does not depend on the ambient parameter space; hence $z|_U$ is continuous, and in particular z is continuous at α_0. $\square$

The hypothesis of Theorem 4.1 constrains jointly the value β_{α_0} and the moduli nearby. Recall that $\alpha \mapsto \beta_\alpha$ is upper semicontinuous at α_0 when

$$\limsup_{\alpha \to \alpha_0} \beta_\alpha \ \leq \ \beta_{\alpha_0},$$

equivalently when $\{\alpha : \beta_\alpha < t\}$ is open for every $t \in \mathbb{R}$ (Folland, 1999). Since $\beta_{\alpha_0} < 1$ is already a hypothesis of Theorem 2.1, upper semicontinuity transfers that strict inequality to a whole neighbourhood and the lim sup condition follows. The implication runs one way only: upper semicontinuity is strictly stronger, since the nearby moduli may have limit superior below 1 and still exceed β_{α_0}.

Corollary 4.2. *If $\beta_\alpha < 1$ for every α and $\alpha \mapsto \beta_\alpha$ is upper semicontinuous, then z is continuous on Ω.*

Proof. Upper semicontinuity gives $\limsup_{\alpha \to \alpha_0} \beta_\alpha \leq \beta_{\alpha_0} < 1$ at every α_0; apply Theorem 4.1. □

These conditions are sufficient, not necessary. The counterexample establishes that the hypotheses of Theorem 2.1 alone do not guarantee continuity; it does not establish that a family violating the lim sup condition must have a discontinuous solution. Deleting the forcing term from (4) makes this plain: the operators $\widetilde{T}\left(x, \frac{1}{n}\right) = \beta_n A_n x$ and $\widetilde{T}(x, 0) = 0$ have $z(1/n) = 0 = z(0)$ for every n, so z is constant and continuous, even though $\beta_{1/n} \to 1$ and upper semicontinuity still fails at $\alpha = 0$.

What the counterexample does locate is the gap in Theorem 2.1. In Section 3 one has $\beta_0 = \mathrm{Lip}(T_0) = 0$ while $\beta_{1/n} = 1 - \frac{1}{n} \to 1$, so that $\limsup_n \beta_{1/n} = 1 > 0 = \beta_0$ and upper semicontinuity fails at $\alpha = 0$: the hypotheses of Theorem 2.1 place no upper restriction on the moduli at nearby parameters, and the construction exploits that freedom.

The discount factor. Let S be a measurable state space, let the feasible sets $\Gamma(s)$ be nonempty, let r be bounded and measurable, and let $Q(\cdot \mid s, a)$ be a transition probability, so that the integral below is well defined and T_β maps $X = B(S)$, the bounded measurable functions with the supremum norm, into itself. Let $\Omega = (0, 1)$ and let $\alpha = \beta$ be the discount factor:

$$(T_\beta v)(s) = \sup_{a \in \Gamma(s)} \left\{ r(s, a) + \beta \int v(s')\, Q(ds' \mid s, a) \right\}. \tag{5}$$

The modulus is exactly β. For the upper bound, $|\sup_a f - \sup_a g| \leq \sup_a |f - g|$ and $Q(\cdot \mid s, a)$ being a probability measure give

$$\left|(T_\beta v)(s) - (T_\beta w)(s)\right| \ \leq \ \beta \sup_{a \in \Gamma(s)} \int \left|v(s') - w(s')\right| Q(ds' \mid s, a) \ \leq \ \beta \|v - w\|_\infty,$$

so $\mathrm{Lip}(T_\beta) \leq \beta$. For the lower bound, test on constants: if $v \equiv c$ then $\int v\, dQ = c$, so $T_\beta v = T_\beta 0 + \beta c$ pointwise and $\|T_\beta v - T_\beta 0\|_\infty = \beta \|v - 0\|_\infty$. Hence $\mathrm{Lip}(T_\beta) = \beta$. The same computation gives continuity in the parameter: for fixed v, $\|T_\beta v - T_{\beta'} v\|_\infty \leq |\beta - \beta'|\, \|v\|_\infty$.

Here $\beta_\alpha = \alpha$ and $\sup_{\alpha \in \Omega} \beta_\alpha = 1$, so Theorem 2.2 does not apply. But $\alpha \mapsto \beta_\alpha$ is the identity, hence continuous, and Corollary 4.2 gives continuity of $\beta \mapsto v_\beta$ on $(0, 1)$. The argument extends to a parameter vector $\alpha = (\beta, \theta)$ in which the discount factor is one coordinate, provided the remaining primitives enter continuously —that is, $\theta \mapsto T(v,(\beta,\theta))$ is continuous for each fixed v— and do not affect the modulus, so that $\beta_\alpha = \beta$ remains a coordinate projection. Within that class of models, the gap in Theorem 2.1 does not bind.

5 The finite-dimensional case

In finite dimensions no hypothesis on the moduli is needed. Two ingredients enter: closed bounded sets are compact, and the family $\{T_\alpha\}_{\alpha\in\Omega}$ is equicontinuous. The second is free. Since $\mathrm{Lip}(T_\alpha) = \beta_\alpha < 1$ for every α, the family is 1-Lipschitz, so $\delta = \varepsilon$ works for every α at once: the common constant is 1, not $\sup_\alpha \beta_\alpha$, and the moduli need not be bounded away from 1 for this. Recall that $\mathcal{F} \subseteq C(K, X)$ is equicontinuous when for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for every $f \in \mathcal{F}$ simultaneously.

Theorem 5.1 (Riesz, 1918)**.** *Let X be a normed space. The closed ball $B[0;1]$ is compact if and only if* $\dim X < \infty$*. Equivalently, every closed bounded subset of X is compact if and only if* $\dim X < \infty$.

Lemma 5.2. *Let (X, d) be a metric space, let $K \subseteq X$ be compact, let $\{T_\alpha\}_{\alpha\in\Omega}$ be equicontinuous, and let $\alpha_n \to \alpha_0$ with $T_{\alpha_n}x \to T_{\alpha_0}x$ for every $x \in K$. Then*

$$\delta_n = \sup_{x\in K} d\left(T_{\alpha_n}x,\ T_{\alpha_0}x\right) \longrightarrow 0.$$

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be a modulus supplied by equicontinuity for $\varepsilon/3$. By compactness $K \subseteq \bigcup_{i=1}^m B(x_i;\delta)$ for finitely many $x_1, \dots, x_m \in K$, and by pointwise convergence at those finitely many points there is N with $d\left(T_{\alpha_n}x_i, T_{\alpha_0}x_i\right) < \varepsilon/3$ for every $i \le m$ and every $n \ge N$. Given $x \in K$, choose i with $d(x, x_i) < \delta$. For $n \ge N$,

$$d\left(T_{\alpha_n}x,\ T_{\alpha_0}x\right) \ \le\ d\left(T_{\alpha_n}x,\ T_{\alpha_n}x_i\right) + d\left(T_{\alpha_n}x_i,\ T_{\alpha_0}x_i\right) + d\left(T_{\alpha_0}x_i,\ T_{\alpha_0}x\right) \ <\ \varepsilon,$$

the outer terms by equicontinuity and the middle one by the choice of N. Hence $\delta_n \le \varepsilon$ for all $n \ge N$.[2] □

The counterexample of Section 3 is the failure of Lemma 5.2 on balls: in ℓ^2 the unit ball is bounded but not compact, no finite net exists, and

$$\sup_{\|x\|\le 1} \left\|T\left(x, \tfrac{1}{n}\right) - T(x,0)\right\| \ \ge\ \left\|T\left(z_n, \tfrac{1}{n}\right)\right\| = \|z_n\| = 1 \qquad \text{for every } n,$$

where $z_n = z(1/n)$, although $T\left(x, \tfrac{1}{n}\right) \to T(x,0)$ for each fixed x.

Theorem 5.3. *Let X be a closed subset of a finite-dimensional normed space, let Ω be a metric space, let $T : X \times \Omega \to X$ be continuous in α, and suppose T_α is a contraction for each $\alpha \in \Omega$, with arbitrary modulus $\beta_\alpha < 1$. Then z is continuous.*

[2]The lemma also follows from Arzelà–Ascoli (see Tao, 2010): the restrictions $T_{\alpha_n}|_K$ form an equicontinuous family, pointwise relatively compact because each orbit $\{T_{\alpha_n}x\}_n$ converges, so some subsequence converges uniformly, and its limit is $T_{\alpha_0}|_K$ by pointwise identification; a contradiction argument then upgrades this to the full sequence. That route verifies the compactness hypothesis by supplying the limit Arzelà–Ascoli exists to produce, and returns a subsequence where the whole sequence is wanted. Compactness of K is used above only to extract a finite δ-net.

Proof. Fix $\alpha_0 \in \Omega$ and write $x^* = z(\alpha_0)$ and $\beta_0 = \mathrm{Lip}(T_{\alpha_0}) < 1$. Let $\alpha_n \to \alpha_0$ and let $r > 0$ be arbitrary. By Theorem 5.1 the set

$$K = B[x^*; r] \cap X$$

is closed and bounded in a finite-dimensional space, hence compact. Continuity of T in α gives $T_{\alpha_n} x \to T_{\alpha_0} x$ for every $x \in K$, and $\{T_\alpha\}$ is equicontinuous because it is 1-Lipschitz; Lemma 5.2 applied to K yields $\delta_n \to 0$. Choose N with $\delta_n \leq (1 - \beta_0)\, r$ for $n \geq N$, which is possible because $\beta_0 < 1$. For such n and every $x \in K$,

$$d\left(T_{\alpha_n} x,\, x^*\right) \;\leq\; d\left(T_{\alpha_n} x,\, T_{\alpha_0} x\right) + d\left(T_{\alpha_0} x,\, T_{\alpha_0} x^*\right) \;\leq\; \delta_n + \beta_0\, r \;\leq\; r,$$

where $T_{\alpha_0} x^* = x^*$ was used. Hence $T_{\alpha_n}(K) \subseteq K$. Since K is closed, nonempty and complete, and $T_{\alpha_n}|_K$ is a contraction, Banach's theorem yields a fixed point of T_{α_n} in K; by global uniqueness of the fixed point of T_{α_n} in X, that point is $z(\alpha_n)$. Therefore $d\left(z(\alpha_n), z(\alpha_0)\right) \leq r$ for all $n \geq N$, and since $r > 0$ was arbitrary, $z(\alpha_n) \to z(\alpha_0)$. □

The estimate never passes through (1): the fixed point is located by trapping it in K, not by bounding its distance to x^*. Accordingly $\beta_\alpha < 1$ is used quantitatively only at α_0, where it opens the gap $(1 - \beta_0)r$ into which δ_n must eventually fit; at every other parameter the contraction property serves only to make $z(\alpha)$ exist and be unique, and the moduli β_{α_n} may approach 1 freely. Compactness supplies the uniformity that Section 4 instead extracts from the moduli. Finite-dimensionality enters only through the compactness of $K = B[x^*; r] \cap X$, so the proof goes through verbatim in any proper metric space, with no linear structure, and locally whenever $z(\alpha_0)$ has a compact neighbourhood.

Compactness of closed balls is also exactly what the mechanism of Section 3 needs to fail: a bounded sequence of fixed points with no convergent subsequence. Theorem 5.1 places that mechanism precisely. Its easy direction rules the failure out when $\dim X < \infty$; its converse says that in every infinite-dimensional normed space the closed unit ball is non-compact, so the room the construction exploits is available throughout. Availability is not occurrence — the truncated family of Section 4 has a continuous solution function in ℓ^2— and the general infinite-dimensional case is not classified here.

6 Conclusion

Theorem 2.1, as stated, is false: Proposition 3.1 exhibits an affine family in ℓ^2 meeting all of its hypotheses —under the joint reading of continuity in the parameter as well as the pointwise one— whose solution function jumps. It is enough for the bound on the moduli to hold near the parameter at which continuity is claimed, $\limsup_{\alpha \to \alpha_0} \beta_\alpha < 1$ (Theorem 4.1). Upper semicontinuity of $\alpha \mapsto \beta_\alpha$ together with $\beta_{\alpha_0} < 1$ implies that condition and is easier to check (Corollary 4.2), but is strictly stronger; neither is necessary for z to be continuous.

This covers the exercise in which the parameter is the discount factor and $\Omega = (0, 1)$, where the supremum of the moduli is one and $\alpha \mapsto \beta_\alpha$ is the identity, and the case of a parameter vector containing the discount factor as a coordinate, provided the remaining primitives enter continuously without affecting the modulus.

The failure also disappears once closed balls are compact: the original statement is then correct without modification (Theorem 5.3). Finite-dimensionality is sufficient for this, but the proof uses only compactness of the relevant closed ball, so the positive result holds in any proper metric space.

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