

Economics 202A: Macroeconomics

Lecture Notes*

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*Lecture notes for Economics 202A (Part I, Macroeconomics), Prof. Jón Steinsson, University of California, Berkeley. Unless identified as reproductions, data figures are schematic reconstructions anchored on values readable off the original plots, not the underlying series.

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1 Growth and labor

1.1 Production, technology, and the labor share

AI and robotics renew an old question: does labor-saving technology make workers worse off? We distinguish **real wages** from the **labor share of income**; one can rise while the other falls.

1.1.1 Production and factor payments

Write $Y_t = F(A_t, K_t, L_t)$, with output Y , capital K , labor L , and technology A . The running example is $Y = AK^\alpha L^{1-\alpha}$.

Definition 1.1 (Returns to scale). f is homogeneous of degree m in its rival inputs if $f(\lambda K, \lambda L, A) = \lambda^m f(K, L, A)$: returns are decreasing for $m < 1$, constant for $m = 1$, and increasing for $m > 1$.

Cobb–Douglas has constant returns in (K, L) . The replication argument assumes efficient plant size, exhausted specialization gains, negligible fixed factors such as land, and negligible net externalities between establishments. Technology is **non-rival**: a second plant uses the same knowledge. Thus constant returns in rival inputs coexist with increasing returns when knowledge is counted too.

Proposition 1.2 (Euler’s theorem). *For differentiable f homogeneous of degree m ,*¹

$$\sum_i f_{x_i}(x) x_i = m f(x). \quad (1)$$

Corollary 1.3 (Competitive factor payments exhaust output). *With constant returns and competitive markets, $w = F_L$, $r = F_K$, and*

$$\begin{aligned} Y &= wL + rK, \\ s_L &\equiv \frac{wL}{Y}, \quad s_K \equiv \frac{rK}{Y}, \\ s_L + s_K &= 1. \end{aligned} \quad (2)$$

Pure profits are zero and firm scale is indeterminate; with identical technologies a representative firm suffices. GDP can be measured as value added, expenditure $C + I + G + X - M$, or income (factor income, mixed income, and net production taxes).

¹Differentiate $f(\lambda x) = \lambda^m f(x)$ with respect to λ and set $\lambda = 1$; see [Acemoglu \[2009, p. 29\]](#).

Definition 1.4 (Elasticity of substitution). Along an isoquant, $0 = F_K dK + F_L dL$, so $MRTS = F_K/F_L$ is the absolute slope. In competitive factor markets, its curvature determines

$$\sigma = \frac{d \log(L/K)}{d \log(F_K/F_L)} = \frac{d \log(L/K)}{d \log(r/w)}. \quad (3)$$

- The isoquant fixes output. A ray from the origin measures L/K ; the tangent measures $-F_K/F_L = -r/w$.
- σ compares their **percentage changes**: when capital becomes relatively expensive, how much does the chosen L/K rise?
- Leontief has $\sigma = 0$, Cobb–Douglas $\sigma = 1$, and linear production $\sigma = \infty$.

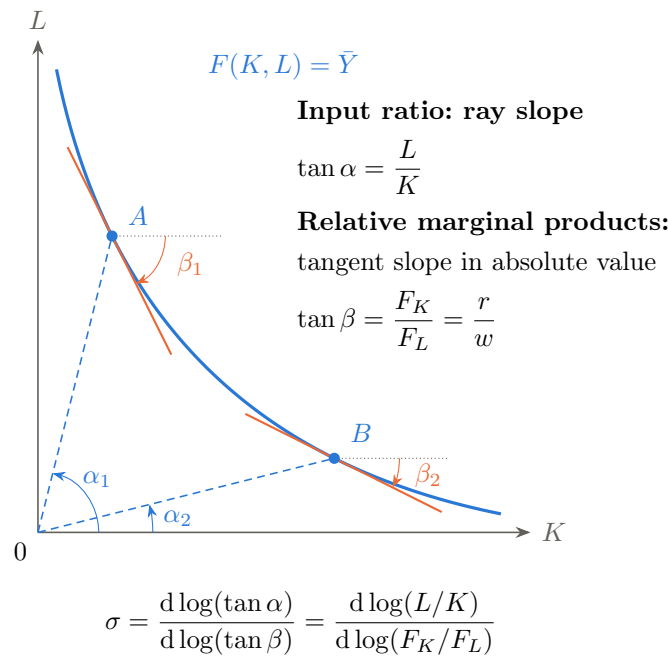


Figure 1: A CES isoquant with $\sigma = 2$. Rays measure input ratios; tangents measure relative marginal products. The elasticity compares proportional changes in these slopes, not the angles; β_i is the positive tangent angle.

Definition 1.5 (Forms of technical change). An X -augmenting technology multiplies input X .

Form	Production	Ratio unchanged when holding fixed
Hicks neutral	$AF(K, L)$	F_K/F_L at a given K/L
Harrod neutral	$F(K, AL)$	$F_K K/(F_L L)$ at a given K/Y
Solow neutral	$F(AK, L)$	$F_K K/(F_L L)$ at a given L/Y

These follow from homogeneity and the chain rule.² Cobb–Douglas satisfies all three:

$$\begin{aligned} AK^\alpha L^{1-\alpha} &= K^\alpha (A^{1/(1-\alpha)} L)^{1-\alpha} \\ &= (A^{1/\alpha} K)^\alpha L^{1-\alpha}. \end{aligned}$$

It cannot distinguish their economic meanings.

1.1.2 Balanced growth and Uzawa

Fact 1.6 (Kaldor). As per capita income rises, K/Y , real interest rates, and factor shares have been approximately stable [Kaldor, 1961; Jones, 2016].

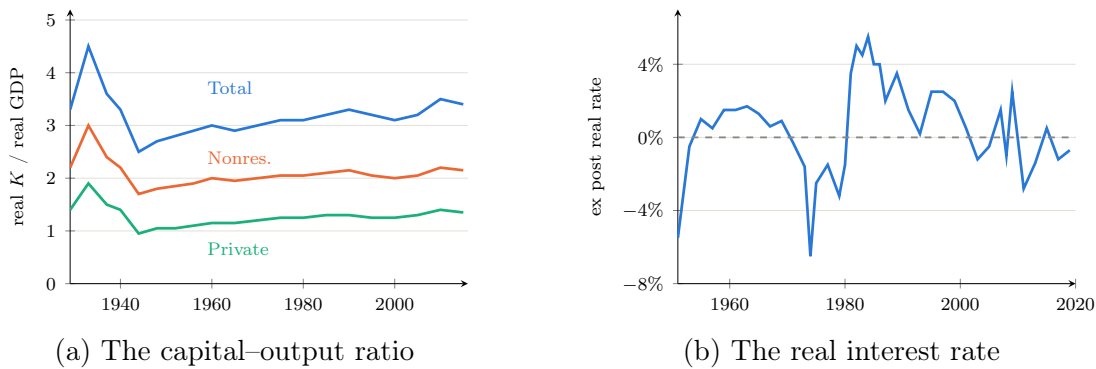


Figure 2: United States. (a) Real capital over real GDP, three definitions of capital. (b) Three-month Treasury bill rate minus CPI inflation. Jones [2016] and FRED.

K/Y compares a stock with an annual flow and has units of years. The interest-rate plot is ex post (nominal rate minus realized inflation); the ex ante rate instead subtracts expected inflation. Cheaper investment goods need not raise real K/Y : the shift toward rapidly depreciating equipment and software can offset the price decline.

Assumption 1.7 (Uzawa environment). F is continuously differentiable and constant returns in (K, L) :

$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A). \quad (4)$$

$$\begin{aligned} Y &= C + I, \\ \dot{K} &= I - \delta K, \\ L(t) &= L(0)e^{nt}. \end{aligned} \quad (5)$$

Definition 1.8 (Balanced growth path). After some T , positive Y , K , and C grow at constant positive rates.

²Under constant returns, marginal products depend only on effective input ratios. For $F(K, AL)$, $K/Y = k/f(k)$ fixes $k = K/(AL)$ under the usual neoclassical conditions, hence fixes the share ratio; the capital-augmenting case is symmetric. Hicks neutrality cancels A directly in the marginal-product ratio.

Theorem 1.9 (Uzawa, 1961). *On a balanced growth path, $g_Y = g_K = g_C$ and technology admits a labor-augmenting representation along that path:*

$$Y(t) = \tilde{F}(K(t), A(t)L(t)),$$

$$\frac{\dot{A}}{A} = g_Y - n. \quad (6)$$

Balanced growth constrains aggregate technical change.³ For a CES technology with $\sigma \neq 1$ and nonvanishing factor shares, balanced growth requires a constant capital-augmenting coefficient. This seems puzzling when inventions improve machines. But a better computer can be interpreted as more computers or as time saved by its user; task models will distinguish these mechanisms. Cobb–Douglas is the special case in which capital augmentation can be rewritten as labor augmentation.

1.1.3 Evidence on wages and shares

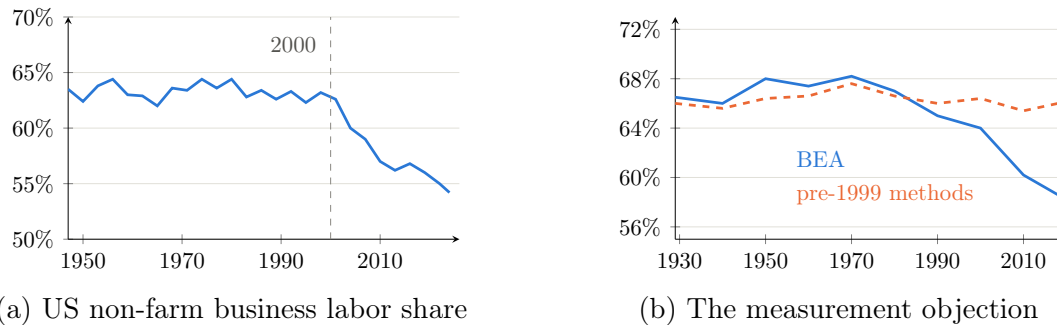


Figure 3: The US labor share. (a) Non-farm business sector, 1947–2024, BLS. (b) Economy-wide BEA measure against pre-1999 methods, after Koh et al. [2020].

The US nonfarm business labor share was broadly stable until about 2000, then fell roughly 9 percentage points. Measurement matters:

- **Intellectual-property investment.** Koh et al. [2020] find the long-run fall disappears under pre-1999 accounting methods. Capitalizing software and R&D raises GDP and measured capital income; their high depreciation means a rising part of gross income replaces worn-out capital. Net domestic product is more relevant for consumption possibilities.

³Proof idea: $I = (g_K + \delta)K$; the positive consumption and investment shares can sum to one at all dates only if their constant growth rates equal g_Y . Rescale by $e^{-g_Y(t-T)}$: capital and output return to their date- T levels, while labor shrinks at rate $g_Y - n$. Effective labor must compensate. This argument concerns the balanced path, not arbitrary input combinations away from it. See Acemoglu [2009, sec. 2.7] and Barro and Sala-i Martin [2004, sec. 1.2.12].

- **Tax-driven reclassification.** [Smith et al. \[2022\]](#) attribute about one-third of the corporate-sector decline to owners reporting labor income as S-corporation profits and high-labor-share firms moving to partnerships. Neither adjustment necessarily changes what workers receive.

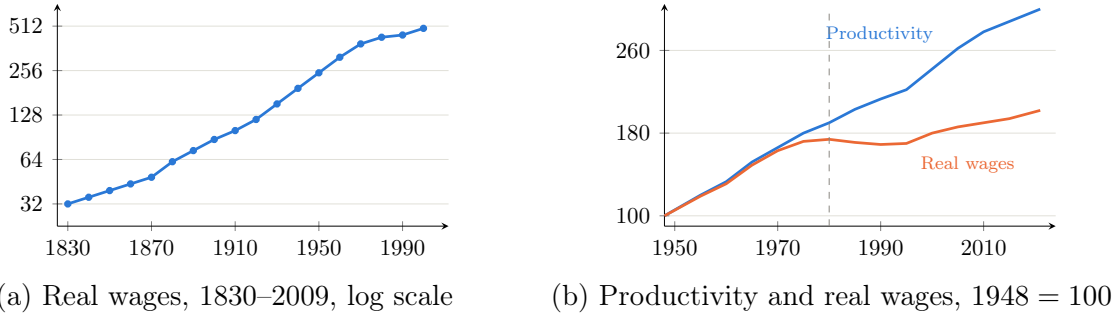


Figure 4: United States. (a) Decadal averages, deflated by the CPI [[Williamson, 1995](#)]; the axis doubles at each tick. (b) Output less depreciation per hour, and hourly compensation of production and non-supervisory workers [[Bivens and Mishel, 2015](#)].

Real wages rose greatly over two centuries of labor-saving innovation. Since around 1980, productivity and compensation of production and nonsupervisory workers have diverged. That gap reflects distribution within labor as well as factor shares and measurement; it is not itself the labor share.

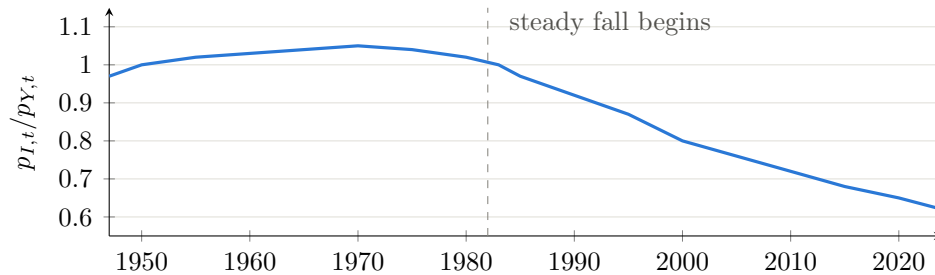


Figure 5: United States, 1947–2024: price of investment goods relative to the price of output, 1950 = 1. Flat until the early 1980s, then down almost 40%.

1.1.4 Cobb–Douglas and CES

For Cobb–Douglas,

$$w = (1 - \alpha)AK^\alpha L^{-\alpha},$$

$$\frac{\partial \log w}{\partial \log A} = 1, \quad \frac{\partial \log w}{\partial \log K} = \alpha. \quad (7)$$

$$s_L = \frac{wL}{Y} = 1 - \alpha. \quad (8)$$

Technology raises wages and both factors' incomes proportionally. To relax constant shares, use the CES function

$$Y_t = \left[\alpha (A_{K,t} K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) (A_{L,t} L_t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}. \quad (9)$$

Differentiating and multiplying each marginal product by its input gives

$$\begin{aligned} w_t &= (1-\alpha) A_{L,t}^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{L_t} \right)^{1/\sigma}, \\ r_t &= \alpha A_{K,t}^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{K_t} \right)^{1/\sigma}. \end{aligned} \quad (10)$$

$$\begin{aligned} s_{L,t} &= (1-\alpha) \left(\frac{A_{L,t} L_t}{Y_t} \right)^{\frac{\sigma-1}{\sigma}}, \\ s_{K,t} &= \alpha \left(\frac{A_{K,t} K_t}{Y_t} \right)^{\frac{\sigma-1}{\sigma}}. \end{aligned} \quad (11)$$

Holding technology and labor fixed,

$$\frac{d \log s_K}{d \log K} = \frac{\sigma-1}{\sigma} (1-s_K) = s_L \frac{\sigma-1}{\sigma}. \quad (12)$$

For **gross substitutes** ($\sigma > 1$), the more abundant factor earns a larger share; for **gross complements** ($\sigma < 1$), its price falls so much that its share declines. At $\sigma = 1$ the effects cancel.

Proposition 1.10 (Capital and wages under CES). *Substituting output into the wage,*

$$w_t = (1-\alpha) A_{L,t}^{\frac{\sigma-1}{\sigma}} \left[\alpha A_{K,t}^{\frac{\sigma-1}{\sigma}} (K_t/L_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) A_{L,t}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}}, \quad (13)$$

so $\partial w / \partial K > 0$ and $\partial w / \partial A_K > 0$ for finite $\sigma > 0$ at positive inputs.

More productive capital raises wages in this model; factor augmentation alone does not capture displacement from tasks.

Corollary 1.11 (Factor shares). *Rearranging capital demand yields*

$$s_K = \alpha^\sigma (A_K/r)^{\sigma-1}, \quad s_L = 1 - s_K. \quad (14)$$

A rise in A_K/r lowers s_L if $\sigma > 1$ and raises it if $\sigma < 1$.

Karabarbounis and Neiman [2014] argue that cheaper investment reduces the cost of capital and, with $\sigma > 1$, the labor share. Estimates below one, such as Oberfield and Raval [2021], reverse that prediction.

1.1.5 Identifying the elasticity

Let $k_t \equiv \log(K_t/L_t)$ and $\omega_t \equiv \log(w_t/r_t)$. Taking the ratio of factor demands and then logs gives

$$\begin{aligned} k_t &= \sigma \omega_t + \epsilon_t, \\ \epsilon_t &= \sigma \log \frac{\alpha}{1-\alpha} + (1-\sigma) \log \frac{A_{L,t}}{A_{K,t}}. \end{aligned} \tag{15}$$

Thus σ is the slope of **relative factor demand**. OLS generally fails because equilibrium prices covary with the technology shock ϵ_t . For illustration, relative supply is

$$k_t = -\gamma \omega_t + u_t, \quad \gamma > 0. \tag{16}$$

Supply shifts u_t trace demand (good variation); technology shifts ϵ_t trace supply (bad variation). An instrument must shift relative supply without shifting technology.⁴

1.1.6 The canonical labor-market model

Replace capital and labor with high- and low-skilled labor, H and L (usually education groups) [Katz and Murphy, 1992; Acemoglu and Autor, 2011]:

$$Y = \left[(A_L L)^{\frac{\sigma-1}{\sigma}} + (A_H H)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}. \tag{17}$$

Proposition 1.12 (Low-skilled wages).

$$w_L = A_L^{\frac{\sigma-1}{\sigma}} \left[A_L^{\frac{\sigma-1}{\sigma}} + A_H^{\frac{\sigma-1}{\sigma}} (H/L)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}}. \tag{18}$$

More H or higher A_H raises w_L ; in the empirically relevant $\sigma > 1$ case, higher A_L also raises it.

Thus skill-biased technical progress does not itself explain falling low-skilled wages.⁵

Definition 1.13 (Skill premium). The skill premium is w_H/w_L , often reported as its logarithm.

Proposition 1.14 (Skill premium). *Taking the ratio of marginal products,*

$$\log \frac{w_H}{w_L} = \frac{\sigma-1}{\sigma} \log \frac{A_H}{A_L} - \frac{1}{\sigma} \log \frac{H}{L}. \tag{19}$$

⁴Proof idea: solve demand and supply jointly. With uncorrelated shocks, OLS is a variance-weighted mixture of the demand slope σ and supply slope $-\gamma$. A valid instrument is correlated with relative prices but orthogonal to the technology shock.

⁵Differentiating gives elasticity $s_H/\sigma > 0$ with respect to either H or A_H . Own-productivity growth has elasticity $(\sigma - s_H)/\sigma$, which can be negative for strong complements. The elasticity is positive in the empirical case $\sigma > 1$.

Education ($H/L \uparrow$) lowers the premium; relative productivity $A_H/A_L \uparrow$ raises it for $\sigma > 1$, but lowers it for $\sigma < 1$. Estimates of σ are about 1.4–2. This is the **race between education and technology** [Tinbergen, 1974; Goldin and Katz, 2008]: rough balance in 1950–80, then technology outpaced the growth in relative skill supply. A rising premium despite rising H/L identifies a shift toward skill demand. Declining real wages for some low-skilled men in 1970–95 motivate task models, although the measured decline depends on the deflator and inclusion of benefits.

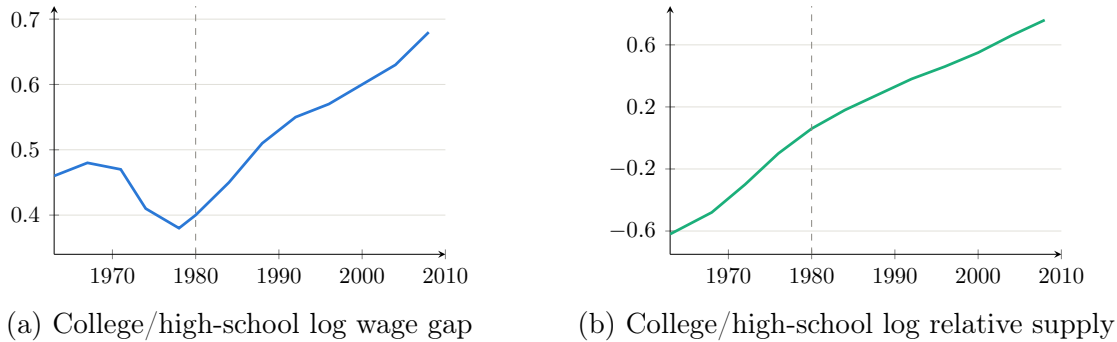
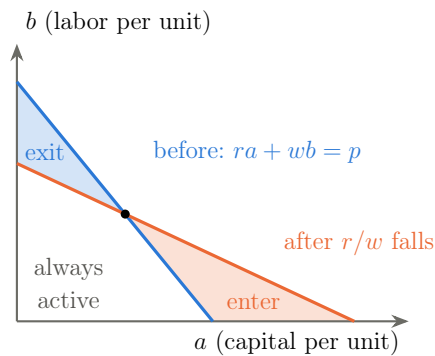


Figure 6: College versus high school, 1963–2008: (a) composition-adjusted log weekly-wage ratio; (b) log relative hours, indexed. After Acemoglu and Autor [2011].

1.1.7 Aggregation

An aggregate production function requires restrictive aggregation conditions (Cambridge Capital Controversy; Harcourt, 1969; Samuelson, 1966). Simplifying assumptions permit analysis, but their limits must remain visible. Aggregate σ includes both substitution within firms and reallocation across firms and industries.⁶



Proposition 1.15 (Houthakker).
Fixed-capacity Leontief plants ($\sigma = 0$), with a suitable Pareto distribution of capacity over input requirements, can generate a Cobb–Douglas aggregate technology ($\sigma_{\text{agg}} = 1$).

Figure 7: Plants below $ra + wb = p$ operate. A fall in r/w reallocates production toward capital-intensive plants; aggregate K/L rises without substitution inside any plant.

⁶Houthakker’s proof integrates output and input requirements over profitable plants; Pareto capacity generates the power functions that yield Cobb–Douglas. See Houthakker [1955].

2 The Task Automation Model

Following [Hassanein and Steinsson \[2026\]](#), task models distinguish **better machines at existing tasks** from **machines taking over workers' tasks**. The second channel can lower wages even as output rises. See also [Zeira \[1998\]](#); [Acemoglu and Autor \[2011\]](#), [Acemoglu and Restrepo \[2018\]](#), and [Jones and Tonetti \[2026\]](#).

2.1 Environment and task allocation

A household supplies L_t inelastically, rents its fixed endowment K_t , and consumes output. A continuum of tasks $\nu \in [0, 1]$ produces the final good:

$$Y_t = \left[\int_0^1 Y_t(\nu)^{\frac{\sigma-1}{\sigma}} d\nu \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 0, \quad (20)$$

$$Y_t(\nu) = \psi_{kt}(\nu)K_t(\nu) + \psi_{lt}(\nu)L_t(\nu). \quad (21)$$

Capital and labor are perfect substitutes **within each task**; σ governs substitution **across tasks**. Tasks are gross complements if $\sigma < 1$, gross substitutes if $\sigma > 1$.⁷ All output is consumed, $Y_t = C_t$. Factor allocations satisfy

$$\begin{aligned} \int_0^1 K_t(\nu) d\nu &= K_t, \\ \int_0^1 L_t(\nu) d\nu &= L_t. \end{aligned} \quad (22)$$

Final-good producer. With the final good as numeraire ($P_t = 1$), solve

$$\max_{\{Y_t(\nu)\}_{\nu \in [0,1]}} \left\{ Y_t - \int_0^1 P_t(\nu) Y_t(\nu) d\nu \right\} \quad \text{subject to (20).}$$

The first-order condition and the implied task demand are

$$P_t(\nu) = \left(\frac{Y_t}{Y_t(\nu)} \right)^{1/\sigma}, \quad (23)$$

$$Y_t(\nu) = P_t(\nu)^{-\sigma} Y_t.$$

$$\frac{Y_t(\nu)}{Y_t(\nu')} = \left[\frac{P_t(\nu)}{P_t(\nu')} \right]^{-\sigma}. \quad (24)$$

⁷The formulas using $1/(\sigma - 1)$ assume $\sigma \neq 1$. At $\sigma = 1$ the unit-measure task aggregator has the Cobb–Douglas limit $\log Y = \int_0^1 \log Y(\nu) d\nu$.

Zero profits imply the ideal price index

$$1 = P_t = \left[\int_0^1 P_t(\nu)^{1-\sigma} d\nu \right]^{\frac{1}{1-\sigma}}. \quad (25)$$

Its derivation follows from zero profits.⁸

Task producers. Competitive firms use the cheaper factor:

$$P_t(\nu) = \min \left\{ \frac{W_t}{\psi_{\ell t}(\nu)}, \frac{R_t}{\psi_{kt}(\nu)} \right\}. \quad (26)$$

Order tasks so that $\psi_{kt}(\nu)/\psi_{\ell t}(\nu)$ decreases in ν : labor has comparative advantage at high-index tasks.

Proposition 2.1 (Smooth automation). *For a continuous relative-productivity schedule and an interior cutoff,*

$$\frac{R_t}{\psi_{kt}(\alpha_t)} = \frac{W_t}{\psi_{\ell t}(\alpha_t)}. \quad (27)$$

Tasks below α_t use capital; tasks above it use labor.

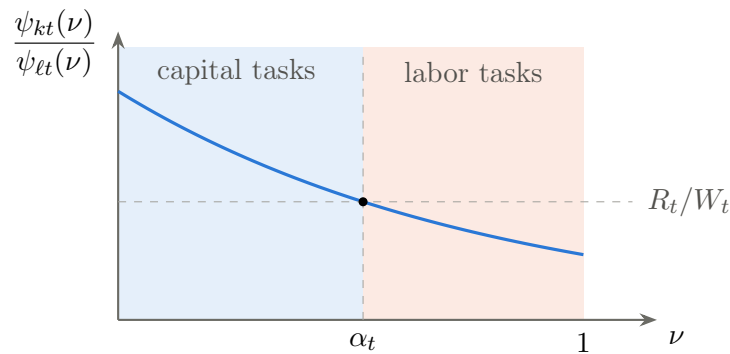


Figure 8: The automation cutoff.

An increase in $\psi_{kt}(\nu)$ lowers capital's unit cost. Its effect depends on where the improvement occurs:

- **Inside the automated region:** lower task prices, pure productivity.
- **Near the cutoff:** lower prices and more automation, displacing labor.
- **Beyond the cutoff:** no immediate effect if capital remains more expensive, though further improvements can eventually automate the task.

⁸Substitute task demand into the expenditure identity and divide both sides by final output. The integral of task prices to the power $1 - \sigma$ must equal one.

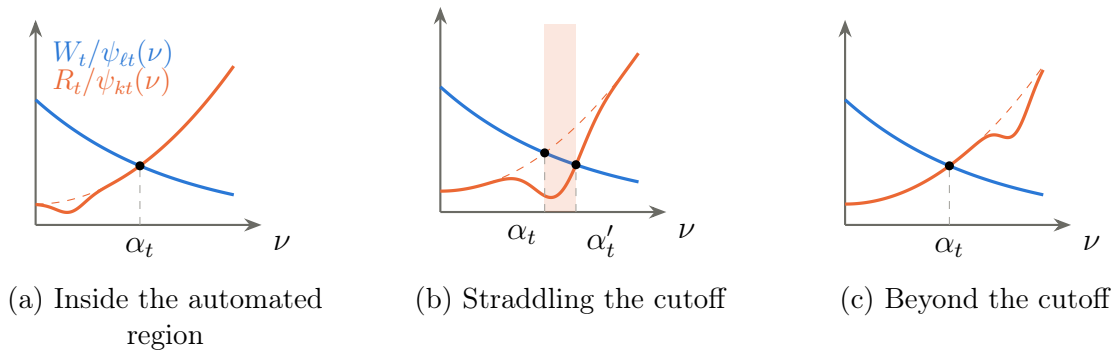


Figure 9: Three improvements in $\psi_{kt}(\nu)$; dashed is capital's old unit cost. (a) Productivity only. (b) Productivity and automation: workers are displaced from $[\alpha_t, \alpha'_t]$. (c) No effect yet.

These diagrams isolate cost changes at given factor prices; equilibrium R_t/W_t also responds through market clearing.

2.2 Uniform productivities and aggregate production

Assumption 2.2 (Uniform productivities).

$$\psi_{kt}(\nu) = \begin{cases} \psi_{kt} & \nu \in [0, \alpha_t], \\ 0 & \nu \in (\alpha_t, 1], \end{cases} \quad (28)$$

$$\psi_{\ell t}(\nu) = \psi_{\ell t} \quad \text{for all } \nu.$$

Now $\alpha_t \in (0, 1)$ is a technological primitive, so automation and productivity improvements ψ_{kt} can be studied separately.

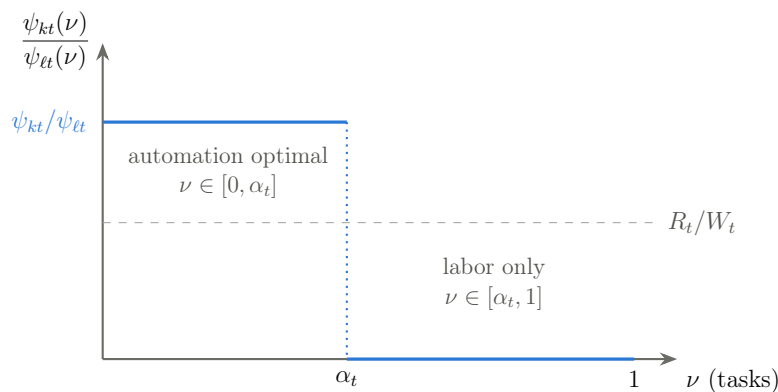


Figure 10: Uniform productivities (Assumption 2.2). To the right of α_t capital has zero productivity, so those tasks can only be done by labor.

With capital employed, $R_t/\psi_{kt} \leq W_t/\psi_{\ell t}$. At equality labor can also perform automatable tasks. We focus on strict inequality, when every feasible task is automated. This requires

$$\frac{K_t}{L_t} > \frac{\alpha_t}{1 - \alpha_t} \frac{\psi_{\ell t}}{\psi_{kt}}.$$

The threshold follows from the allocation at equal costs.⁹

For a labor task, $L_t(\nu) = Y_t(\nu)/\psi_{\ell t}$; substituting task demand and price gives

$$\begin{aligned} L_t(\nu) &= \begin{cases} \psi_{\ell t}^{\sigma-1} W_t^{-\sigma} Y_t & \nu > \alpha_t, \\ 0 & \nu < \alpha_t, \end{cases} \\ K_t(\nu) &= \begin{cases} \psi_{kt}^{\sigma-1} R_t^{-\sigma} Y_t & \nu < \alpha_t, \\ 0 & \nu > \alpha_t. \end{cases} \end{aligned} \quad (29)$$

Integrating constant demands over their task intervals,

$$\begin{aligned} L_t &= (1 - \alpha_t) \psi_{\ell t}^{\sigma-1} W_t^{-\sigma} Y_t, \\ K_t &= \alpha_t \psi_{kt}^{\sigma-1} R_t^{-\sigma} Y_t. \end{aligned} \quad (30)$$

Solve for factor prices:

$$\begin{aligned} W_t &= (1 - \alpha_t)^{1/\sigma} \psi_{\ell t}^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{L_t} \right)^{1/\sigma}, \\ R_t &= \alpha_t^{1/\sigma} \psi_{kt}^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{K_t} \right)^{1/\sigma}. \end{aligned} \quad (31)$$

The relative wage is

$$\frac{W_t}{R_t} = \left(\frac{1 - \alpha_t}{\alpha_t} \right)^{1/\sigma} \left(\frac{\psi_{kt}}{\psi_{\ell t}} \right)^{\frac{1-\sigma}{\sigma}} \left(\frac{K_t}{L_t} \right)^{1/\sigma}. \quad (32)$$

Proposition 2.3 (Aggregate CES representation).

$$Y_t = \left[(A_{kt} K_t)^{\frac{\sigma-1}{\sigma}} + (A_{\ell t} L_t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \quad (33)$$

$$A_{kt} = \alpha_t^{\frac{1}{\sigma-1}} \psi_{kt}, \quad A_{\ell t} = (1 - \alpha_t)^{\frac{1}{\sigma-1}} \psi_{\ell t}, \quad (34)$$

⁹ At equal costs every task has the same output. Assign capital to all α_t feasible tasks and labor to the remainder to obtain the threshold. Below it, some labor remains in automatable tasks.

$$\begin{aligned}
s_{kt} &= \frac{R_t K_t}{Y_t} = \left(\frac{A_{kt} K_t}{Y_t} \right)^{\frac{\sigma-1}{\sigma}}, \\
s_{\ell t} &= \frac{W_t L_t}{Y_t} = \left(\frac{A_{\ell t} L_t}{Y_t} \right)^{\frac{\sigma-1}{\sigma}}, \\
s_{kt} + s_{\ell t} &= 1.
\end{aligned} \tag{35}$$

Derivation: insert task costs into the price index,

$$1 = \alpha_t (R_t / \psi_{kt})^{1-\sigma} + (1 - \alpha_t) (W_t / \psi_{\ell t})^{1-\sigma},$$

Substitute (31) and multiply by $Y_t^{(\sigma-1)/\sigma}$:

$$Y_t^{\frac{\sigma-1}{\sigma}} = \alpha_t^{1/\sigma} (\psi_{kt} K_t)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_t)^{1/\sigma} (\psi_{\ell t} L_t)^{\frac{\sigma-1}{\sigma}}.$$

The definitions of A_{kt} and $A_{\ell t}$ then give (33). Unlike traditional models, A_{kt} and $A_{\ell t}$ are **composite coefficients**. With $\sigma < 1$, raising ψ_{kt} raises A_{kt} , while automation lowers A_{kt} and raises $A_{\ell t}$. A better computer can therefore affect both aggregate coefficients.

2.3 Weak links and Baumol's cost disease

Cotton spinning, computing, or AI can transform particular tasks without proportionately transforming total output. For $\sigma < 1$,

$$Y_t^{\frac{\sigma-1}{\sigma}} = (A_{kt} K_t)^{\frac{\sigma-1}{\sigma}} + (A_{\ell t} L_t)^{\frac{\sigma-1}{\sigma}}. \tag{36}$$

The exponent is negative, so the capital term vanishes as $A_{kt} K_t \rightarrow \infty$:

$$\boxed{\tilde{Y}_t \equiv \lim_{A_{kt} K_t \rightarrow \infty} Y_t = A_{\ell t} L_t.} \tag{37}$$

With complementary tasks, output is limited by tasks that remain hard to improve. This bound holds at fixed α_t , $\psi_{\ell t}$, and L_t ; automation can relax it. The remaining gain from perfecting existing machines is

$$\frac{\tilde{Y}_t}{Y_t} = (1 - s_{kt})^{-\frac{\sigma}{1-\sigma}} \simeq 1 + \frac{\sigma}{1 - \sigma} s_{kt}. \tag{38}$$

The approximation applies to small s_{kt} .¹⁰ For $\sigma \geq 1$ there is no finite ceiling as capital productivity tends to infinity with a positive share of automated tasks.

¹⁰Substitute the labor share from (35), use shares summing to one, and take a first-order Taylor expansion around a zero capital share.

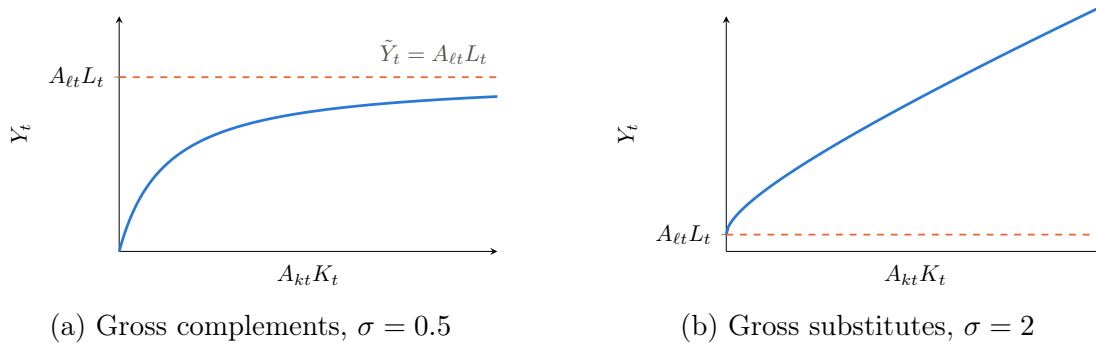


Figure 11: Aggregate output as capital becomes infinitely productive, holding $A_{\ell t} L_t = 1$. Left: output asymptotes to the **weak-links** bound $\tilde{Y}_t = A_{\ell t} L_t$. Right: output grows without bound and $A_{\ell t} L_t$ places no ceiling on it.

Remark 2.4 (Baumol’s cost disease). From task demand, relative expenditure is

$$\frac{P_t(\nu)Y_t(\nu)}{P_t(\nu')Y_t(\nu')} = \left[\frac{P_t(\nu)}{P_t(\nu')} \right]^{1-\sigma}.$$

When $\sigma < 1$, a task’s relative price falls faster than its quantity rises: spending shifts toward slow-productivity tasks [Baumol, 1967].

2.4 Technical change and productivity

Log-differentiate aggregate output; the elasticities equal factor shares:

$$\frac{d \ln Y_t}{dt} = s_{kt} \left(\frac{d \ln A_{kt}}{dt} + \frac{d \ln K_t}{dt} \right) + s_{\ell t} \left(\frac{d \ln A_{\ell t}}{dt} + \frac{d \ln L_t}{dt} \right). \quad (39)$$

The Solow residual subtracts input growth:

$$d \ln \text{TFP}_t = d \ln Y_t - s_{kt} d \ln K_t - s_{\ell t} d \ln L_t.$$

From (34),

$$\begin{aligned} d \ln A_{kt} &= d \ln \psi_{kt} + \frac{d \ln \alpha_t}{\sigma - 1}, \\ d \ln A_{\ell t} &= d \ln \psi_{\ell t} - \frac{\alpha_t}{(\sigma - 1)(1 - \alpha_t)} d \ln \alpha_t. \end{aligned} \quad (40)$$

Substitution separates productivity from automation:

$$\begin{aligned} d \ln \text{TFP}_t &= s_{kt} d \ln \psi_{kt} + s_{\ell t} d \ln \psi_{\ell t} \\ &\quad + \frac{1}{\sigma - 1} \left(s_{kt} - \frac{\alpha_t}{1 - \alpha_t} s_{\ell t} \right) d \ln \alpha_t. \end{aligned}$$

Market clearing writes the shares in terms of task unit costs:

$$s_{kt} = \alpha_t \left(\frac{R_t}{\psi_{kt}} \right)^{1-\sigma},$$

$$s_{\ell t} = (1 - \alpha_t) \left(\frac{W_t}{\psi_{\ell t}} \right)^{1-\sigma}.$$

Define their unit-cost gap in the relevant power as

$$\Delta_t \equiv \left(\frac{R_t}{\psi_{kt}} \right)^{1-\sigma} - \left(\frac{W_t}{\psi_{\ell t}} \right)^{1-\sigma}.$$

Proposition 2.5 (Three sources of productivity growth).

$$\frac{d \ln \text{TFP}_t}{dt} = \underbrace{s_{\ell t} \frac{d \ln \psi_{\ell t}}{dt}}_{\text{better labor}} + \underbrace{s_{kt} \frac{d \ln \psi_{kt}}{dt}}_{\text{better machines}} + \underbrace{\frac{\Delta_t}{\sigma - 1} \dot{\alpha}_t}_{\text{automation}}. \quad (41)$$

Holding K_t , L_t , ψ_{kt} , and $\psi_{\ell t}$ fixed,

$$q_{\alpha,t} \equiv \frac{\partial \ln Y_t}{\partial \ln \alpha_t} = \frac{\alpha_t \Delta_t}{\sigma - 1} \geq 0. \quad (42)$$

Automation raises productivity because capital is cheaper on the newly automated tasks.¹¹ With a smooth cutoff, marginal unit costs coincide, so the direct automation term is zero to first order.¹² Yet automation lets more tasks benefit from future machine improvements. Jones and Tonetti [2026] attribute most TFP growth to that channel; without continued automation and with gross complements, automated tasks' spending share would shrink.

2.5 Wages, rental rates, and factor shares

Taking logs of the wage equation gives

$$d \ln W_t = \frac{1}{\sigma} \left[-\frac{\alpha_t}{1 - \alpha_t} d \ln \alpha_t + (\sigma - 1) d \ln \psi_{\ell t} + d \ln Y_t - d \ln L_t \right].$$

Output elasticities equal shares:

¹¹Since $R_t/\psi_{kt} \leq W_t/\psi_{\ell t}$, Δ_t and $\sigma - 1$ have the same sign; their ratio is nonnegative.

¹²Apply Leibniz's rule to each region's productivity integral. The two boundary terms cancel because capital and labor have equal unit costs at the cutoff; productivity improvements inside each region still contribute.

$$\frac{\partial \ln Y_t}{\partial \ln \psi_{kt}} = \frac{\partial \ln Y_t}{\partial \ln K_t} = s_{kt},$$

and analogously for labor.

Proposition 2.6 (Factor-price elasticities). *Each row varies its argument while holding the other primitives fixed.*

x	$\frac{\partial \ln W_t}{\partial \ln x}$	$\frac{\partial \ln R_t}{\partial \ln x}$
α_t	$\frac{1}{\sigma} \left(q_{\alpha,t} - \frac{\alpha_t}{1 - \alpha_t} \right)$	$\frac{1 + q_{\alpha,t}}{\sigma} > 0$
ψ_{kt}	$\frac{s_{kt}}{\sigma} > 0$	$\frac{\sigma - s_{\ell t}}{\sigma}$
$\psi_{\ell t}$	$\frac{\sigma - s_{kt}}{\sigma}$	$\frac{s_{\ell t}}{\sigma} > 0$
K_t	$\frac{s_{kt}}{\sigma} > 0$	$-\frac{s_{\ell t}}{\sigma} < 0$
L_t	$-\frac{s_{kt}}{\sigma} < 0$	$\frac{s_{\ell t}}{\sigma} > 0$

(43)

The wage effects have three interpretations:

- **Productivity:** better machines at existing tasks raise output and wages, without removing a task from labor.
- **Displacement:** automation also concentrates workers on fewer tasks. **Automation lowers wages when displacement exceeds the productivity gain:**

$$q_{\alpha,t} < \frac{\alpha_t}{1 - \alpha_t}.$$

- **Abundance:** more labor lowers its wage; better labor combines a positive direct productivity effect with greater effective supply. Its wage effect is negative if $\sigma < s_{kt}$. Capital deepening instead makes labor relatively scarce and raises its wage.

For example, a faster mixer helps electricians on complementary tasks; a machine that also installs wiring removes their task demand.

Subtracting factor-price elasticities,

$$\frac{\partial \ln(W_t/R_t)}{\partial \ln \alpha_t} = -\frac{1}{\sigma(1 - \alpha_t)} < 0, \quad (44)$$

$$\begin{aligned}\frac{\partial \ln(W_t/R_t)}{\partial \ln(\psi_{kt}/\psi_{\ell t})} &= \frac{1-\sigma}{\sigma}, \\ \frac{\partial \ln(W_t/R_t)}{\partial \ln(K_t/L_t)} &= \frac{1}{\sigma}.\end{aligned}\tag{45}$$

Automation always reduces the relative wage; capital deepening always raises it. With complements, better machines also raise the relative wage through Baumol's mechanism.

Proposition 2.7 (Labor-share elasticities).

$$\begin{aligned}\frac{\partial \ln s_{\ell t}}{\partial \ln \alpha_t} &= -\frac{s_{kt}}{\sigma(1-\alpha_t)} < 0, \\ \frac{\partial \ln s_{\ell t}}{\partial \ln \psi_{kt}} &= -\frac{\partial \ln s_{\ell t}}{\partial \ln \psi_{\ell t}} = s_{kt} \frac{1-\sigma}{\sigma}, \\ \frac{\partial \ln s_{\ell t}}{\partial \ln(K_t/L_t)} &= s_{kt} \frac{1-\sigma}{\sigma}.\end{aligned}\tag{46}$$

These follow by log-differentiating the labor share.¹³ **Automation always reduces labor's share; better machines raise it when tasks are complements.** These are distinct changes in technology.

2.6 Stable factor shares and Uzawa

For complementary tasks ($\sigma < 1$), better machines and automation can offset one another. From (34),

$$\frac{d \ln A_{kt}}{dt} = \frac{d \ln \psi_{kt}}{dt} - \frac{1}{1-\sigma} \frac{d \ln \alpha_t}{dt}.\tag{47}$$

The aggregate capital coefficient stays constant when

$$\frac{d \ln \alpha_t}{dt} = (1-\sigma) \frac{d \ln \psi_{kt}}{dt}.\tag{48}$$

Meanwhile automation raises $A_{\ell t}$ even with constant $\psi_{\ell t}$. **Machine innovation can be labor augmenting in aggregate** [Jones and Liu, 2024].¹⁴

¹³Subtract the output elasticity from the wage elasticity, adding one for a change in labor supply. The same result follows from (35) and (40).

¹⁴This is a local balancing condition. On a fixed unit interval $\alpha_t \leq 1$, automation cannot grow proportionally at a positive rate forever. A full long-run model must specify task creation or the approach to complete automation.

The capital share: automation versus machine efficiency. Suppose machines are produced one-for-one from output and used up, so $R_t = 1$ and gross output satisfies $Y_t = C_t + K_t$. Market clearing gives

$$s_{kt} = \frac{K_t}{Y_t} = \int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1} d\nu = \boxed{\frac{\alpha_t}{Z_{kt}}}. \quad (49)$$

Machine efficiency index. Z_{kt} summarizes productivity across automated tasks:

$$Z_{kt} \equiv \left[\frac{1}{\alpha_t} \int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1} d\nu \right]^{-1} = \alpha_t A_{kt}^{1-\sigma}. \quad (50)$$

With uniform productivity,

$$Z_{kt} = \psi_{kt}^{1-\sigma} \implies s_{kt} = \alpha_t \psi_{kt}^{\sigma-1}.$$

For $\sigma < 1$:

- **More automated tasks:** higher α_t at a given Z_{kt} raises the capital share, because machines perform a larger part of production.
- **Better machines:** higher productivity at a fixed task set raises Z_{kt} and lowers the capital share. Existing machine tasks become cheaper; with complements, spending shifts toward the remaining labor tasks.
- **Stable share:** equal proportional growth of α_t and Z_{kt} keeps their ratio fixed. For example, raising both by 10% leaves s_{kt} unchanged.

The accounting identity and its uniform-productivity special case are

$$\begin{aligned} d \ln s_{kt} &= d \ln \alpha_t - d \ln Z_{kt}, \\ d \ln Z_{kt} &= (1 - \sigma) d \ln \psi_{kt}. \end{aligned}$$

Thus the condition for a stable capital share is exactly (48).¹⁵

¹⁵With heterogeneous productivity, changing the automated set may also change its average productivity and hence Z_{kt} . The first comparison holds that index fixed; the accounting identity includes both changes. The equality in (50) uses the CES aggregate of machine productivities over the automated region.

3 Capital Accumulation and Growth: the Solow Model

- **Steady frontier growth:** US GDP per person grew about 2% annually in 1870–2014, from roughly \$3,000 to over \$50,000 [Jones, 2016]. A straight line in logs means constant proportional growth.
- **Recent and uneven growth:** English income fluctuated without sustained growth before 1800; linear scales hide those large fluctuations [Clark, 2010]. Japan and China later caught up with the US, while Argentina and South Africa fell behind [Jones, 2021]. A flat income ratio to the US means equal growth, not stagnation.

Four questions motivate growth theory: what sustains frontier growth, why countries lag behind, why growth began, and why it was absent before the Industrial Revolution. Solow asks whether capital accumulation can explain sustained growth [Solow, 1956].¹⁶

3.1 The model

Definition 3.1 (Neoclassical production function). $Y = F(K, L)$ is **neoclassical** if it satisfies

- (i) **Constant returns to scale:**

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

- (ii) **Positive and diminishing marginal products:**

$$F_K, F_L > 0, \quad F_{KK}, F_{LL} < 0.$$

- (iii) **Essential inputs:**

$$F(0, L) = F(K, 0) = 0.$$

- (iv) **Inada conditions**, holding the other input fixed [Inada, 1964]:

$$\begin{aligned} \lim_{K \rightarrow 0^+} F_K &= \lim_{L \rightarrow 0^+} F_L = \infty, \\ \lim_{K \rightarrow \infty} F_K &= \lim_{L \rightarrow \infty} F_L = 0. \end{aligned}$$

¹⁶The view that capital accumulation drives growth was conventional in the 1950s [Easterly, 2002]. For analyses of neoclassical growth, see Romer [2019, Ch. 1.1–1.6], Acemoglu [2009], and Barro and Sala-i Martin [2004]. Very early historical income estimates are especially uncertain.

Cobb–Douglas, $F(K, L) = AK^\alpha L^{1-\alpha}$ with $0 < \alpha < 1$, satisfies these conditions.¹⁷

Assumption 3.2. Time is continuous. Technology is labor augmenting (Harrod neutral), and a constant fraction s of output is saved:

$$Y(t) = F[K(t), A(t)L(t)], \quad (51)$$

$$Y(t) = C(t) + I(t), \quad (52)$$

$$\begin{aligned} I(t) &= sY(t), \\ K'(t) &= I(t) - \delta K(t). \end{aligned} \quad (53)$$

Population and technology grow exogenously:

$$\begin{aligned} L'(t) &= nL(t), \\ A'(t) &= gA(t). \end{aligned} \quad (54)$$

Thus $L(t) = L(0)e^{nt}$ and $A(t) = A(0)e^{gt}$. Assume $0 < s < 1$, $n + g + \delta > 0$, and positive initial conditions $K(0), A(0), L(0)$.

Solow isolates a mechanism; realistic policy counterfactuals generally require more structure. Neither labor nor technology is chosen within it.

When A or L grows, normalize to obtain a steady state. Capital and output per **effective worker** are

$$\begin{aligned} k &\equiv \frac{K}{AL}, \\ y &\equiv \frac{Y}{AL} = f(k), \quad f(k) \equiv F(k, 1). \end{aligned}$$

The last equality follows from constant returns to scale.

Proposition 3.3 (Law of motion).

$$k'(t) = \underbrace{s f(k(t))}_{\text{actual investment}} - \underbrace{(n + g + \delta) k(t)}_{\text{break-even investment}}. \quad (55)$$

Differentiate capital per effective worker:

$$\begin{aligned} k' &= \frac{K'}{AL} - (n + g)k \\ &= sf(k) - \delta k - (n + g)k. \end{aligned}$$

¹⁷Linear production lacks diminishing returns and essential inputs; Leontief is not differentiable at its kink. For CES with finite $\sigma > 0$, both Inada limits hold only when $\sigma = 1$: with $\sigma > 1$ the marginal product does not vanish at infinity; with $\sigma < 1$ it is finite at zero.

Break-even investment replaces depreciation (δ) and equips the additional effective labor ($n + g$).

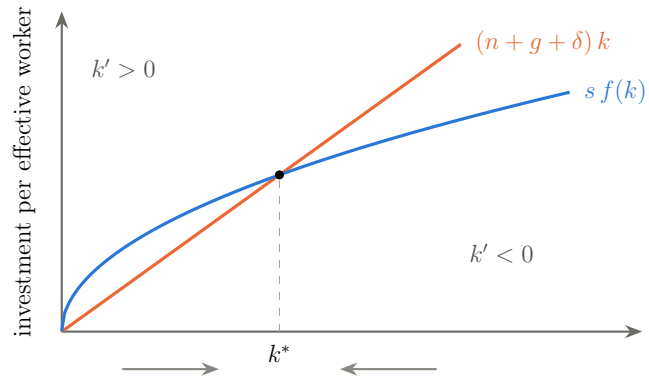


Figure 12: Actual against break-even investment. $f(0) = 0$ makes the two curves meet at the origin; the Inada condition $f'(0) = \infty$ makes $sf(k)$ steeper there; $f'(\infty) = 0$ makes it flatter eventually; and $f'' < 0$ makes them cross exactly once at $k^* > 0$. Arrows show the direction of motion, so k^* is stable and attracts every path with $k(0) > 0$.

Proposition 3.4 (Steady state and balanced growth). *Under Definition 3.1, the unique positive steady state solves*

$$sf(k^*) = (n + g + \delta)k^*.$$

From any $k(0) > 0$, capital per effective worker converges to k^ . On the balanced growth path:*

- K and Y grow at $n + g$;
- K/L and Y/L grow at g .

Why convergence? Strict concavity makes $f(k)/k$ decreasing, and Inada gives its limits $+\infty$ and 0 . Thus investment crosses break-even investment once: $k' > 0$ below k^* and $k' < 0$ above it.¹⁸

Fixed technology and population.

Let $A, L > 0$ be constant and $\delta > 0$. With Hicks-neutral technology,

$$\begin{aligned} Y &= AK^\alpha L^{1-\alpha}, \\ \dot{K} &= sAK^\alpha L^{1-\alpha} - \delta K. \end{aligned}$$

¹⁸Concavity and $f(0) = 0$ give $f'(k) < f(k)/k$, hence $\frac{d}{dk}[f(k)/k] = [kf'(k) - f(k)]/k^2 < 0$. The law of motion then yields a monotone, bounded path converging to k^* . At that point $K = ALk^*$ and $Y = ALf(k^*)$ grow with AL .

Here A multiplies output.¹⁹ Setting $\dot{K} = 0$ and dividing by $(K^*)^\alpha$ at a positive steady state gives

$$(K^*)^{1-\alpha} = \frac{sA}{\delta} L^{1-\alpha},$$

$$K^* = L \left(\frac{sA}{\delta} \right)^{1/(1-\alpha)}.$$

Below K^* capital grows; above it capital falls. This unique positive steady state attracts every $K(0) > 0$. Zero capital is also a boundary steady state.

3.2 The two lessons

- **Capital accumulation cannot sustain growth in living standards.** Diminishing returns eventually exhaust growth from capital deepening. If $g = 0$, long-run growth in Y/L is zero; sustained growth requires technological progress, which Solow leaves unexplained.
- **Convergence is to each country's own long-run path.** The growth rate

$$\frac{k'}{k} = \frac{sf(k)}{k} - (n + g + \delta)$$

decreases with k : capital grows faster when it is scarce relative to its destination. For $f(k) = k^\alpha$,

$$k^* = \left(\frac{s}{n + g + \delta} \right)^{1/(1-\alpha)}, \quad (56)$$

$$(K/L)^*(t) = A(t)k^*.$$

Absolute convergence compares initial income with growth without controls: countries sharing A, s, n, g, δ and technology approach the same path. **Conditional convergence** holds these determinants fixed; different fundamentals allow permanently different income levels. For $g = 0$, the examples show current income $\rightarrow$ long-run income:

	Poor country	Rich country
Same fundamentals	10 $\rightarrow$ 60	30 $\rightarrow$ 60
Different saving	10 $\rightarrow$ 10	30 $\rightarrow$ 60

The initially richer country can grow faster if the poor country is already at its own steady state.

¹⁹The corresponding labor-augmenting index is $A^{1/(1-\alpha)}$. Thus effective labor in the earlier normalization is $A^{1/(1-\alpha)}L$.

Remark 3.5 (Comparative statics). A permanent rise in s shifts $sf(k)$ up, so k^* rises and Y/L grows faster than g during the transition, then returns to g . A rise in the savings rate has a **level** effect on income per person, not a **growth** effect. The mirror case is the destruction of half the capital stock: output falls on impact, growth is fast during the recovery, and the long run is untouched. Post-war Germany and Japan are the standard illustration.

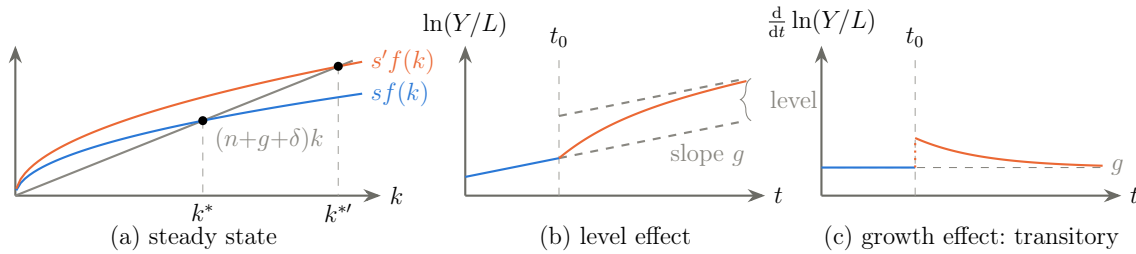


Figure 13: Permanent rise in the saving rate at t_0 . (a) $sf(k)$ shifts up and k^* rises. (b) $\ln(Y/L)$ converges to a parallel balanced growth path: a level effect. (c) The growth rate jumps at t_0 and decays back to g , locally at rate $\lambda = (1 - \alpha)(n + g + \delta)$ near the new path.

For Cobb–Douglas, the income path is

$$(Y/L)^* = A(t) \left(\frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha}},$$

$$\frac{\partial \ln(Y/L)^*}{\partial \ln s} = \frac{\alpha}{1 - \alpha}.$$

Thus higher n or δ lowers the income level; higher A raises it proportionally. Holding g fixed, none changes long-run growth per worker.

Proposition 3.6 (Saving and the golden rule). *Higher saving raises k^* and $y^* = f(k^*)$. Define capital's output elasticity:*

$$\alpha_K(k) \equiv \frac{f'(k)k}{f(k)}.$$

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{n + g + \delta - sf'(k^*)} > 0,$$

$$\frac{\partial \log y^*}{\partial \log s} = \frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)}. \quad (57)$$

Steady-state consumption is output net of break-even investment:

$$c^* = f(k^*) - (n + g + \delta)k^*.$$

Therefore,

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n + g + \delta)] \frac{\partial k^*}{\partial s}. \quad (58)$$

The **golden rule** maximizes steady-state consumption:

$$f'(k_{GR}) = n + g + \delta.$$

Higher saving raises c^* below k_{GR} and lowers it above k_{GR} .²⁰

More output need not mean more consumption. Above the golden rule, extra capital produces less than the investment needed to maintain it. Today capital is fixed, so raising s always reduces consumption, $c = (1 - s)f(k)$; the golden-rule comparison is after capital adjusts.

Proposition 3.7 (Transition dynamics). *Capital and output per effective worker grow at rates*

$$g_k \equiv \frac{k'}{k} = \frac{sf(k)}{k} - (n + g + \delta), \quad (59)$$

$$g_y \equiv \frac{y'}{y} = \alpha_K(k)g_k. \quad (60)$$

Here $sf(k)/k$ decreases from $+\infty$ to 0 as k rises; α_K is the output elasticity of capital (its income share under competition).

Divide (55) by k , then differentiate $y = f(k)$ to obtain these two rates. Since $Y/L = Ay$, income per worker grows at

$$\frac{d \log(Y/L)}{dt} = g + \alpha_K(k)g_k.$$

At k^* , only technological growth g remains.

²⁰Differentiate $sf(k^*) = (n+g+\delta)k^*$ to obtain (57); its denominator is positive because $f'(k^*) < f(k^*)/k^*$. Then differentiate c^* . Strict concavity makes the golden rule the unique consumption maximum.

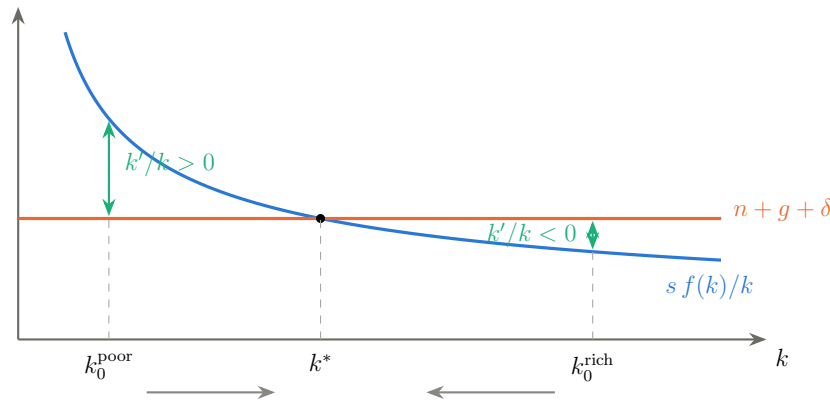


Figure 14: Transition dynamics. The vertical gap $sf(k)/k - (n + g + \delta)$ is g_k : positive below k^* , negative above it, and zero at k^* . With the same fundamentals, lower initial k means faster capital growth. As k approaches k^* from below, growth slows to zero; sustained income growth per worker comes from technology, g .

3.3 Convergence and divergence in the data

β -convergence means that lower initial income predicts faster subsequent growth. **σ -convergence** means that the cross-country standard deviation of log income falls: incomes become less dispersed. Catch-up need not reduce dispersion if new shocks push countries apart.

There are two tests of β -convergence:

- **Absolute (unconditional):** poorer countries grow faster **without controls**, as predicted for countries sharing a long-run path.
- **Conditional:** poorer countries grow faster **holding long-run determinants fixed**.

A poor country already on its own path need not catch a richer one.

Checking σ -convergence. For the same N countries, compute the dispersion of comparable log incomes per worker:

$$\overline{\log q_t} = \frac{1}{N} \sum_{i=1}^N \log q_{it},$$

$$D_t = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (\log q_{it} - \overline{\log q_t})^2}.$$

Compare D_t over time: $D_t < D_{t-T}$ means σ -convergence over that interval; an increase means divergence.

Measurement. Distinguish the unit in which income is measured:

$$\begin{aligned} y_i(t) &\equiv \frac{Y_i(t)}{A_i(t)L_i(t)} && \text{per effective worker,} \\ q_i(t) &\equiv \frac{Y_i(t)}{L_i(t)} = A_i(t)y_i(t) && \text{per worker,} \\ \frac{Y_i(t)}{N_i(t)} &= q_i(t) \frac{L_i(t)}{N_i(t)} && \text{per person.} \end{aligned}$$

Income per effective worker approaches a constant; income per worker approaches the growing path

$$q_i^*(t) = A_i(t)y_i^*.$$

The gap to that path. Define

$$x_i(t) \equiv \log \frac{q_i(t)}{q_i^*(t)}.$$

With common convergence speed λ and technology growth g , Section 3.4 implies, locally,

$$\begin{aligned} x_i(t) &\simeq e^{-\lambda T} x_i(t - T), \\ \log q_i^*(t) &= \log q_i^*(t - T) + gT. \end{aligned}$$

Define the average catch-up coefficient over T years:

$$\gamma_T \equiv \frac{1 - e^{-\lambda T}}{T}.$$

Substituting the definition of x_i and rearranging gives

$$\frac{\log q_i(t) - \log q_i(t - T)}{T} \simeq g + \gamma_T [\log q_i^*(t - T) - \log q_i(t - T)].$$

Growth combines technology and catch-up. Representing the unobserved destination term by an intercept, controls and a residual gives [Barro and Sala-i Martin, 1991]:

$$\frac{1}{T} \log \left(\frac{q_{i,t}}{q_{i,t-T}} \right) = a - \frac{1 - e^{-\lambda T}}{T} \log q_{i,t-T} + \text{controls} + u_i. \quad (61)$$

With controls fixed, the negative coefficient means poorer countries grow faster. Dividing by T gives annual average log growth; u_i is unexplained growth. The coefficient reflects both speed λ and duration T .

Evidence across countries: history, tests, recent changes.

- **Long-run divergence.** Urbanization suggests early divergence [Acemoglu et al., 2005]. Using a \$250 PPP subsistence floor (1985 dollars), Pritchett [1997] estimates that the richest-to-poorest ratio rose from about 9 in 1870 to 45 in 1990. Different long-run paths allow this despite conditional convergence.
- **Early tests.** For sixteen industrialized countries in 1870–1979, Baumol [1986] found nearly complete catch-up: a coefficient near -1 on initial log income in a cumulative-growth regression. Selecting eventual rich countries favours this result; understating initial income also overstates measured growth [De Long, 1988].
- **Broader samples.** Absolute convergence appears within the OECD and regions of one country, but not clearly worldwide in 1960–2011 [Jones, 2016; Barro and Sala-i Martin, 1991]. Controlling for investment, population growth and schooling yields conditional convergence in Mankiw et al. [1992].²¹ These controls neither remove all productivity differences nor establish causality.
- **Since about 2000:** poorer groups grew faster and richer groups slower, producing broader absolute convergence [Kremer et al., 2021]. Investment, population growth and other fundamentals became more similar; why they changed remains open.

From countries to people: poverty and inequality. Country averages, poverty and global inequality answer different questions.

- **Whom do we count?** In 1970–2000, weighting countries by population turns divergence across countries into catch-up across people. National means hide internal inequality. Since about 1980, falling *between-country* inequality has been partly offset by rising *within-country* inequality.
- **Poverty.** Combining national means and household surveys, Sala-i Martin [2006] finds lower global inequality and poverty in 1970–2000: both poverty rates and headcounts fell across several fixed thresholds. Africa was the exception; its growth recovery after 2000 qualifies that earlier pessimism [Archibong et al., 2021].
- **Inequality.** The World Inequality Database (WID) adds tax records to better capture top incomes. Its estimates show inequality rising in 1820–1910, staying high, then declining after 2000. In 2020 the top 10% received about 55% of income; the bottom 50%, 7% [Chancel and Piketty, 2021]. $T10/B50$ compares these groups' mean incomes; the Gini measures relative dispersion (lower means less inequality).
- **Measurement matters.** Tax coverage is sparse in Africa and covers only 0.5% of tax units in China and 2% in India; surveys miss poor incomes too. Using GDP and household final consumption expenditure (HFCE) adjustments, Pinkovskiy et al.

²¹For their 98-country non-oil sample, the coefficient on initial log income in cumulative 1960–1985 growth changes from 0.094 without controls to -0.141 with saving and population growth, and -0.289 after adding schooling. The corresponding conditional annual convergence speeds are 0.006 and 0.014. OECD convergence is visible even without controls.

[2024]'s preferred estimates show a larger inequality decline than WID. All methods find falling poverty, but different levels: correcting the mean need not correct each group's income.

3.4 Speed of convergence

Linearize the law of motion. Write

$$k' = h(k) \equiv sf(k) - (n + g + \delta)k.$$

Since $h(k^*) = 0$, a first-order expansion gives

$$k' \simeq h'(k^*)(k - k^*).$$

Using the steady-state condition,

$$\begin{aligned} \lambda \equiv -h'(k^*) &= (n + g + \delta) - sf'(k^*) \\ &= [1 - \alpha_K(k^*)](n + g + \delta) > 0. \end{aligned} \tag{62}$$

Proposition 3.8 (Convergence rate). *Locally, capital and output gaps decay at the same exponential rate.*²²

$$\begin{aligned} k(t) - k^* &\simeq e^{-\lambda t}[k(0) - k^*], \\ \text{half-life} &= \frac{\log 2}{\lambda}. \end{aligned} \tag{63}$$

Speed measures gap closure. Since $q = Ay$ and $q^* = Ay^*$,

$$\begin{aligned} \log \frac{q}{q^*} &= \log \frac{y}{y^*}, \\ \frac{d \log q}{dt} &\simeq \underbrace{g}_{\text{technology}} + \underbrace{\lambda \log \frac{q^*}{q}}_{\text{catch-up}}. \end{aligned}$$

Technological progress is permanent; catch-up fades. A higher capital elasticity weakens diminishing returns and slows convergence. For Cobb–Douglas, saving changes the destination without changing λ .

²²Solve $\Delta' = -\lambda\Delta$ for $\Delta = k - k^*$. The approximation $y - y^* \simeq f'(k^*)\Delta$ gives the same decay rate for output. Log gaps inherit it as well. Closing a fraction p takes $-\log(1-p)/\lambda$; all these formulas are local.

Recovering speed from the data. The coefficient b on initial log income in (61) satisfies

$$b = -\frac{1 - e^{-\lambda T}}{T},$$

$$\lambda = -\frac{\log(1 + bT)}{T}.$$

The inverse formula recovers the annual speed if the controls adequately capture differences in destinations.

Remark 3.9 (Model versus estimates).

- **The model converges too quickly.** The standard calibration predicts an annual speed of 5.3%, with a half-life of about 13 years.²³
- **Empirical catch-up is slower.** A benchmark speed of 2% implies a half-life of about 35 years; closing 90% of the gap takes about 115 years [Barro and Sala-i Martin, 1991, 1992].

Raising the capital elasticity to about 0.75, possibly by including human capital, brings the model closer to the data [Mankiw et al., 1992]. Estimates vary with the sample and controls.

²³The calibration sets $g = 0.02$, $n = 0.01$, $\delta = 0.05$ and $\alpha_K = 1/3$.

4 The Ramsey–Cass–Koopmans Model

Ramsey keeps Solow’s technology but makes saving a household choice. Lifetime utility makes welfare explicit; expected future income and returns determine consumption today. Long-run growth still comes from exogenous technological progress [Romer, 2019, Ch. 2.1–2.7].

4.1 Environment

Assumption 4.1 (The Ramsey economy).

- (i) **Technology.** Production is neoclassical (Definition 3.1), with **no depreciation** ($\delta = 0$):

$$Y = F(K, AL),$$

$$A' = gA,$$

$$L' = nL,$$

$$K' = Y - CL.$$

Assume $n + g > 0$. Here C is consumption *per person*; output is consumed or invested.

- (ii) **Firms.** Many identical competitive firms hire labor and rent capital; A is free to all of them. Output is the numeraire.
- (iii) **Households.** H identical, infinitely lived households. Each has $N_h \equiv L/H$ members, who grow at rate n and supply one unit of labor each, and starts with capital $K(0)/H$.

Resource constraint. Define quantities per effective worker:

$$k \equiv \frac{K}{AL},$$

$$y \equiv \frac{Y}{AL} = f(k),$$

$$c \equiv \frac{C}{A}.$$

Differentiating the capital ratio and substituting the resource constraint,

$$\begin{aligned} k' &= \frac{K'}{AL} - (n + g)k \\ &= \frac{Y - CL}{AL} - (n + g)k \\ &= f(k) - c - (n + g)k. \end{aligned} \tag{64}$$

The term $(n + g)k$ comes from growth in effective labor, not depreciation.

Proposition 4.2 (Factor prices). *With competition and $\delta = 0$,*

$$\begin{aligned} r &= f'(k), \\ w &\equiv \frac{W}{A} = f(k) - kf'(k), \end{aligned} \tag{65}$$

where W is the wage per worker and w the wage per effective worker.

Differentiate $Y = ALf(k)$ with respect to each input:

$$\begin{aligned} F_K &= f'(k), \\ F_L &= A[f(k) - kf'(k)]. \end{aligned}$$

The factor payments exhaust output, $rk + w = f(k)$, so profits are zero. Common technology and factor prices determine a common capital–labor ratio; constant returns leave firm size indeterminate, allowing a **price-taking representative firm**. Households likewise take prices as given.²⁴

Preferences. Each household values the consumption of all its members:

$$\begin{aligned} U &= \int_0^\infty e^{-\rho t} u(C) N_h dt, \\ u(C) &= \frac{C^{1-\theta} - 1}{1 - \theta}, \quad \theta > 0. \end{aligned} \tag{66}$$

with $u = \log C$ when $\theta = 1$. $\rho > 0$ is impatience and $1/\theta$ is the **elasticity of intertemporal substitution** (EIS).

²⁴For $Y = \tilde{A}K^\alpha L^{1-\alpha}$, $K/L = [\alpha/(1-\alpha)]W/r$, so $Y = \tilde{A}[\alpha W/((1-\alpha)r)]^\alpha L$ is linear in firm size L . With labor-augmenting A , $\tilde{A} = A^{1-\alpha}$.

Household flow budget. Let K_h be the household's **net assets** (capital plus lending minus borrowing; negative means debt), with $K_h(0) = K(0)/H$:

$$\underbrace{K'_h}_{\text{saving}} = \underbrace{rK_h}_{\text{asset income}} + \underbrace{WN_h}_{\text{labor income}} - \underbrace{CN_h}_{\text{consumption}}. \quad (67)$$

Lemma 4.3 (Budget per effective member). *Let $k_h \equiv K_h/(AN_h)$. Then*

$$k'_h = (r - n - g)k_h + w - c. \quad (68)$$

This follows by differentiating $K_h/(AN_h)$: growth in the number of effective members dilutes wealth at rate $n + g$.

With a time-varying interest rate, define cumulative interest:

$$R(t) \equiv \int_0^t r(\tau) d\tau.$$

Then $e^{-R(t)}$ is the date-zero price of output delivered at t .

Definition 4.4 (No-Ponzi-game condition).

$$\lim_{t \rightarrow \infty} e^{-R(t)} K_h(t) \geq 0. \quad (69)$$

Equivalently, in units per effective member,

$$\lim_{t \rightarrow \infty} e^{-R(t) + (n+g)t} k_h(t) \geq 0.$$

Borrowing is allowed, but debt cannot be rolled over forever: spending must be backed by initial wealth or labor income.

Proposition 4.5 (Intertemporal budget constraint). *Under (69),*

$$\int_0^\infty e^{-R(t) + (n+g)t} c(t) dt \leq k_h(0) + \int_0^\infty e^{-R(t) + (n+g)t} w(t) dt, \quad (70)$$

with equality if and only if (69) holds with equality (given finite present-value labor income).

Present-value consumption cannot exceed initial wealth plus present-value labor income.²⁵

²⁵Multiply the flow budget by e^{-R} : $d(e^{-R}K_h)/dt = e^{-R}(W - C)N_h$. Integrate to T , let $T \rightarrow \infty$, apply no-Ponzi, and divide by $A(0)N_h(0)$; use $CN_h = A(0)N_h(0)e^{(n+g)t}c$. No-Ponzi rules out a negative terminal wealth value; optimality below also rules out a positive unused value.

Lemma 4.6 (Preferences per effective worker). *The effective discount rate is*

$$\beta \equiv \rho - n - (1 - \theta)g.$$

Assume $\rho > n$ and $\beta > 0$. Substitute the growth paths:

$$\begin{aligned} C(t) &= A(0)e^{gt}c(t), \\ N_h(t) &= \frac{L(0)}{H}e^{nt}. \end{aligned}$$

Up to a positive multiplier and a constant, utility becomes

$$\int_0^\infty e^{-\beta t} \frac{c^{1-\theta}}{1-\theta} dt. \quad (71)$$

For $\theta = 1$, use $\log c$ and $\beta = \rho - n$.²⁶

4.2 The household's problem

By Lemmas 4.3 and 4.6, the household solves

$$\begin{aligned} \max_{\{c(t)>0\}} \quad & \int_0^\infty e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \\ \text{subject to} \quad & k'_h = (r - n - g)k_h + w - c, \\ & k_h(0) \text{ given,} \\ & \text{the no-Ponzi condition (69).} \end{aligned} \quad (72)$$

taking the paths of r and w as given. Consumption c is the **control**; wealth k_h is the **state**.

First-order condition. A multiplier λ_0 on the lifetime budget gives

$$e^{-\beta t} c(t)^{-\theta} = \lambda_0 e^{-R(t)+(n+g)t}.$$

Take logs and then differentiate with respect to time:

$$\begin{aligned} -\beta t - \theta \log c(t) &= \log \lambda_0 - R(t) + (n + g)t, \\ -\beta - \theta \frac{c'}{c} &= -r + n + g. \end{aligned}$$

²⁶For $\theta \neq 1$, the multiplier is $B = A(0)^{1-\theta} L(0)/H$; normalized utility subtracts the path-independent constant $L(0)/[H(1-\theta)(\rho-n)]$. For log utility, $\log C = \log A(0) + gt + \log c$; the first two terms likewise do not affect the chosen path.

Rearranging gives the consumption Euler equation.²⁷

Proposition 4.7 (Household optimum). *The optimal path satisfies the **Euler equation***

$$\begin{aligned}\frac{c'}{c} &= \frac{r - \rho - \theta g}{\theta} && \text{per effective worker,} \\ \frac{C'}{C} &= \frac{r - \rho}{\theta} && \text{per person.}\end{aligned}\tag{73}$$

Optimality also requires zero unused wealth in present value:

$$\lim_{t \rightarrow \infty} e^{-R(t) + (n+g)t} k_h(t) = 0.$$

Therefore (70) holds with equality.

Economic meaning. Consumption per person rises if $r > \rho$, is constant if $r = \rho$ and falls if $r < \rho$. A larger θ makes it less responsive to $r - \rho$. **The Euler equation fixes consumption growth; the lifetime budget fixes its level.** Income and wealth affect the level even though they do not appear in its growth equation.

4.3 General equilibrium

Households are identical and own all capital, so $k_h = k$, and firms pay $r = f'(k)$.

Definition 4.8 (Competitive equilibrium). Households optimise at given prices, firms pay marginal products, and markets clear. Then

$$\frac{c'}{c} = \frac{f'(k) - \rho - \theta g}{\theta},\tag{74}$$

$$k' = f(k) - c - (n + g)k,\tag{75}$$

Initial capital $k(0)$ is given; the terminal condition is

$$\lim_{t \rightarrow \infty} e^{-R(t) + (n+g)t} k(t) = 0.$$

²⁷The equivalent current-value Hamiltonian and its conditions are

$$\begin{aligned}\mathcal{H} &= \frac{c^{1-\theta}}{1-\theta} + \mu[(r - n - g)k_h + w - c], \\ \mu &= c^{-\theta}, \\ \frac{\mu'}{\mu} &= \beta + n + g - r = \rho + \theta g - r.\end{aligned}$$

Transversality, $\lim_{t \rightarrow \infty} e^{-\beta t} \mu k_h = 0$, gives zero terminal present-value wealth. For the maximum principle and concavity-based sufficiency, see [Acemoglu \[2009, Ch. 7\]](#), [Barro and Sala-i Martin \[2004, App. A.3\]](#), and [Obstfeld \[1992\]](#).

Prices are given by (65).

Capital is inherited and cannot jump. **Initial consumption is chosen**, so it jumps on news; the terminal condition pins it down.

Proposition 4.9 (Steady state and balanced growth). *The unique positive steady state satisfies*

$$\begin{aligned} f'(k^*) &= \rho + \theta g, \\ c^* &= f(k^*) - (n + g)k^*, \\ s^* &= \frac{(n + g)k^*}{f(k^*)}. \end{aligned} \tag{76}$$

and $k^* < k_{GR}$. On the balanced growth path, K , Y and CL grow at $n + g$; K/L , Y/L and C grow at g .

Set $c' = k' = 0$ to obtain (76). The steady-state return exceeds the golden-rule return:

$$f'(k^*) - f'(k_{GR}) = \rho + \theta g - (n + g) = \beta > 0.$$

Since marginal returns decrease, $k^* < k_{GR}$.²⁸ **The golden rule maximizes steady-state consumption; Ramsey values consumption along the entire path.** Accumulation sacrifices consumption today, so patient but discounting households stop short of k_{GR} .

Preferences affect levels and transition dynamics, not long-run growth. Population growth n does not enter $f'(k^*) = \rho + \theta g$ because utility weights all family members. Higher n leaves k^* unchanged, raises the investment share s^* , and lowers c^* . Without that family-size weight, the condition would be $f'(k^*) = \rho + n + \theta g$, and higher n would lower k^* .

4.4 The phase diagram

The diagram (Figure 15) plots (k, c) per effective worker.

- (1) $c' = 0$: the orange vertical line $k = k^*$. To its left $f'(k)$ is higher and c rises; to its right c falls.
- (2) $k' = 0$: the blue curve is

$$c = f(k) - (n + g)k.$$

Below it k rises; above it k falls. The curve peaks at k_{GR} and hits zero at $\bar{k}$, where $f(\bar{k}) = (n + g)\bar{k}$.

²⁸Inada and strictly decreasing f' imply uniqueness. Concavity gives $c^* > k^*[f'(k^*) - n - g] = \beta k^* > 0$. Constant k, y, c give balanced growth by multiplying by A or AL .

- (3) **Steady state:** $E = (k^*, c^*)$, left of the peak since $k^* < k_{GR}$.
- (4) **Stable (saddle) path:** for each k , one initial c leads to E (green). Other choices violate feasibility or optimality.

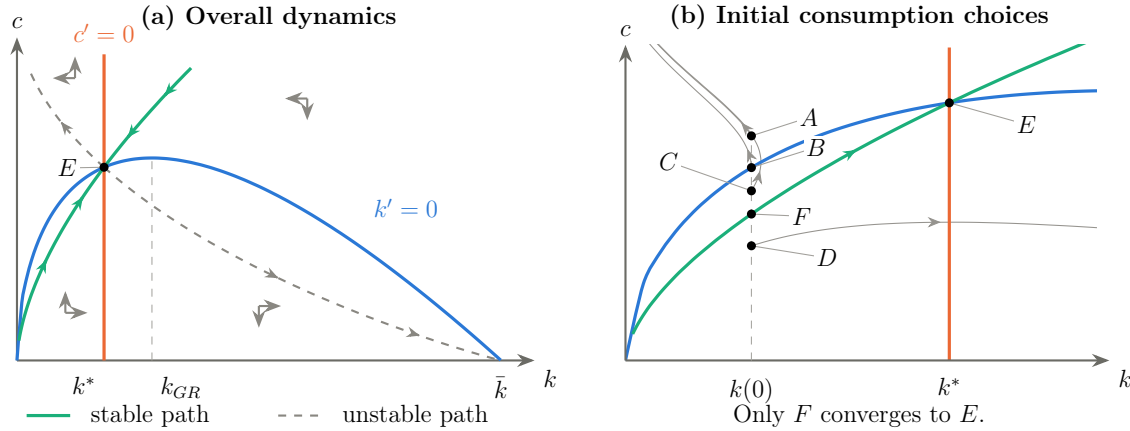


Figure 15: The Ramsey phase diagram. (a) Below the blue curve capital rises; left of the orange line consumption rises. (b) Consumption choices at $k(0)$. Only F is on the stable path to E . A, B, C exhaust capital; D violates transversality.

Proposition 4.10 (The saddle path is the equilibrium). *For each $k(0) > 0$, one initial consumption satisfies feasibility and transversality. The associated policy $c_s(k)$ leads monotonically to E .²⁹*

For $k(0) < k^*$ in Figure 15:

- **Too much consumption (A, B, C):** capital eventually runs out while consumption remains positive, violating feasibility.
- **Too little (D):** capital accumulates toward $\bar{k}$ and consumption tends to zero. Since $f'(\bar{k}) < n + g$, discounted remaining wealth explodes: the household could consume more, violating optimality.
- **On the stable path (F):** both variables approach E ; bounded k and $r - (n + g) \rightarrow \beta > 0$ imply transversality.

²⁹Locally, with $x = k - k^*$ and $z = c - c^*$, the Jacobian for (z, x) is

$$J = \begin{pmatrix} 0 & c^* f''(k^*)/\theta \\ -1 & \beta \end{pmatrix}.$$

Its determinant is negative, so the eigenvalues have opposite signs:

$$\mu_{1,2} = \frac{\beta \mp \sqrt{\beta^2 - 4c^* f''(k^*)/\theta}}{2}.$$

Initial consumption eliminates the unstable component; the remaining gap decays at rate $-\mu_1 > 0$. The phase-diagram argument gives the global selection; see **Romer** [2019, Ch. 2].

Capital and consumption move together on the stable path, so $c'_s(k) > 0$. Above k^* , both fall toward E .

4.5 A fall in the discount rate

Start at the steady state. Households unexpectedly become permanently more patient: ρ falls (still with $\rho > n$, $\beta > 0$). Nothing else changes. Note that ρ is impatience, not the market return r .

1. Long run. Households accumulate until the return is lower:

$$\rho \downarrow \implies f'(k^*) \downarrow \implies k^* \uparrow.$$

The $c' = 0$ line moves right; the $k' = 0$ curve does not move, since technology is unchanged.

- 2. Impact.** k cannot jump, so output and r are unchanged. Consumption drops from E to A on the new stable path, and investment rises.
- 3. Transition.** Capital and output rise, consumption recovers along the stable path toward E' , and $r = f'(k)$ falls to its new level.
- 4. Outcome.** Capital, output and consumption per effective worker are permanently higher; c rises because the new k^* is still below k_{GR} . Output per person grows faster than g during the transition, then at g on a higher path.

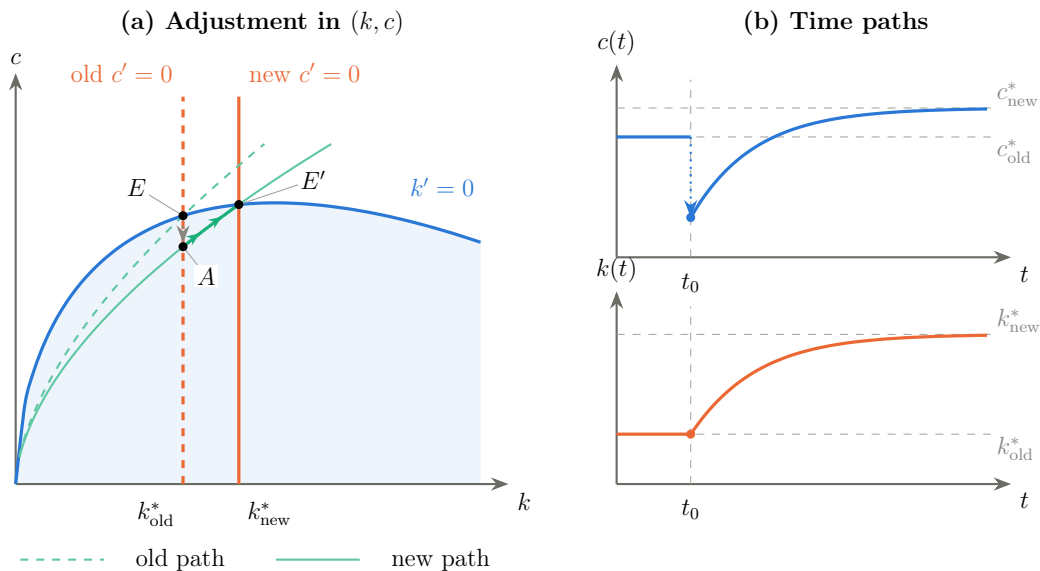


Figure 16: An unexpected permanent fall in impatience ρ . (a) Consumption falls from E to A at fixed capital; both then rise toward E' . Dashed green: old stable path. (b) Consumption drops, then rises above its old level; capital rises continuously. Both variables are per effective worker.

4.6 θ and the stable manifold

For CRRA utility, curvature and intertemporal substitution satisfy

$$\theta = -\frac{Cu''(C)}{u'(C)},$$

$$\text{EIS} = \frac{1}{\theta}.$$

A larger θ means stronger smoothing: consumption growth reacts less to the return $r - \rho$.³⁰

Proposition 4.11 (Curvature and the stable path). *Set $g = 0$ to hold E fixed. Higher θ flattens the stable path near E and slows convergence.*³¹

Below k^* , stronger smoothing means sacrificing less consumption today: c_0 is closer to c^* , leaving less for accumulation. The high- θ path lies closer to the $k' = 0$ curve in Figure 17.

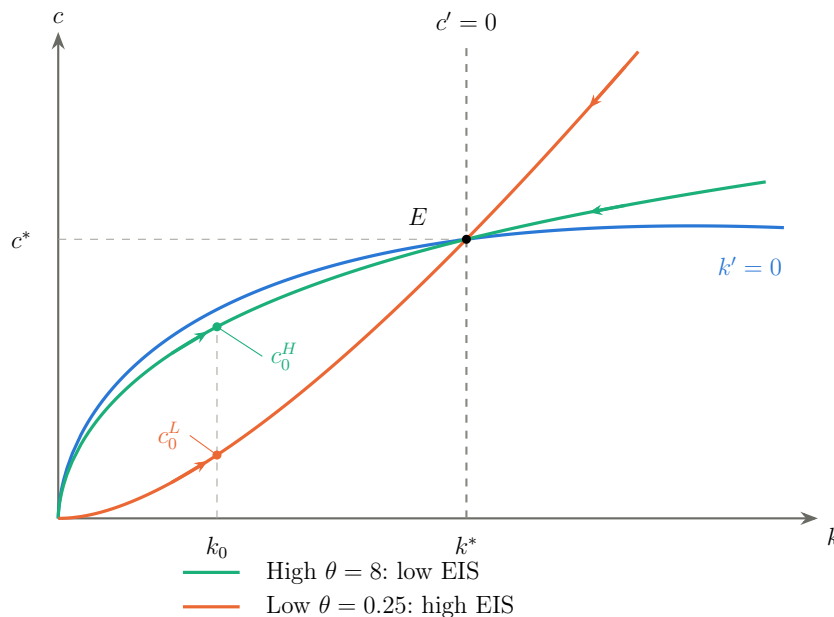


Figure 17: With $g = 0$, high θ (green) gives a flatter stable path than low θ (orange): at $k_0 < k^*$, $c_0^H > c_0^L$. Both converge to E ; blue is $k' = 0$.

³⁰In two periods, $u'(C_0) = D(1+r)u'(C_1)$ gives $C_1/C_0 = [D(1+r)]^{1/\theta}$, so $d \log(C_1/C_0)/d \log(1+r) = 1/\theta$.

³¹Its local slope is

$$m = \frac{1}{2} \left[\beta + \sqrt{\beta^2 - 4c^* f''(k^*)/\theta} \right].$$

It decreases with θ ; the convergence rate is $m - \beta$. As $\theta \rightarrow \infty$, $m \rightarrow \beta = f'(k^*) - n$, the local slope of the $k' = 0$ curve. With $g > 0$, the steady state also moves: $dk^*/d\theta = g/f''(k^*) < 0$, so a common- E comparison requires $g = 0$.

4.7 The saving rate along the transition

Ramsey determines saving from optimal consumption:

$$s = 1 - \frac{c}{f(k)}.$$

Below k^* , income and consumption both rise. Two forces determine their relative growth:

- **Substitution effect.** Scarce capital earns a high return, encouraging saving early. As k rises and $r = f'(k)$ falls, future consumption becomes relatively more expensive, tending to lower s .
- **Income effect.** Expecting higher future income, households smooth consumption by consuming a large share of low current income. As income catches up, this force tends to raise s .

Preferences and technology determine which force dominates. Differentiating the saving rate, with $y = f(k)$, gives

$$s' = (1 - s) \left(\frac{y'}{y} - \frac{c'}{c} \right).$$

Thus s rises if output grows faster than consumption and falls otherwise. On the balanced growth path,

$$s = s^* = \frac{(n + g)k^*}{f(k^*)} :$$

saving finances the investment needed to keep k constant.

Cobb–Douglas benchmark. For $f(k) = k^\alpha$, saving is monotone along the stable path. Starting below k^* , define

$$D \equiv \alpha(n + g) - \frac{\rho + \theta g}{\theta}.$$

- $D > 0$: the saving rate rises.
- $D = 0$: it stays constant at $s = 1/\theta$.
- $D < 0$: it falls.

Starting above k^* reverses the time directions.³² Empirically, saving rises slightly with development, making near-constant saving a plausible benchmark.

4.8 Welfare

Proposition 4.12 (Efficiency). *Competitive equilibrium is **Pareto efficient**. With identical households it solves*

$$\begin{aligned} \max_{\{c(t) > 0\}} \quad & \int_0^\infty e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \\ \text{subject to} \quad & k' = f(k) - c - (n+g)k, \\ & k(0) \text{ given,} \\ & k(t) \geq 0. \end{aligned}$$

For log utility use $\log c$; optimality also imposes

$$\lim_{t \rightarrow \infty} e^{-\beta t} c(t)^{-\theta} k(t) = 0.$$

The planner and market choose the same path.³³

The private return equals the social return, $r = f'(k)$. Competitive and complete markets, no externalities, optimization and property rights make the first welfare theorem apply here. Raising steady-state consumption toward the golden rule need not improve lifetime welfare, because reaching it costs consumption during the transition.

³²For a constant saving rate, $c = (1-s)k^\alpha$. Compare its growth with the Euler equation:

$$\begin{aligned} \frac{c'}{c} &= \alpha \frac{k'}{k} = \alpha [s k^{\alpha-1} - (n+g)], \\ \frac{c'}{c} &= \frac{\alpha k^{\alpha-1} - \rho - \theta g}{\theta}. \end{aligned}$$

Matching coefficients gives $s = 1/\theta$ and $D = 0$. For the monotonicity cases, use the phase diagram for c/k^α and k ; see Barro and Sala-i Martin [2004, Sec. 2.6.4].

³³The planner's Hamiltonian gives $\mu = c^{-\theta}$ and $\mu'/\mu = \beta + n + g - f'(k)$: the same Euler, resource and terminal conditions as the market. Concavity makes them sufficient.

5 Growth Accounting

Growth accounting decomposes output growth into contributions from capital, labor and a productivity residual [Abramovitz, 1956; Solow, 1957]. It does not explain why inputs or productivity grow.

5.1 The Solow residual

Start from the production function of the Solow and Ramsey models,

$$Y(t) = F[K(t), A(t)L(t)], \quad (77)$$

with F differentiable and A labor-augmenting. Write $g_X \equiv X'(t)/X(t)$ for the growth rate of any variable X , and F_1, F_2 for the partial derivatives of F with respect to its first and second arguments. Since A and L enter only through the product AL ,

$$\frac{\partial Y}{\partial K} = F_1, \quad \frac{\partial Y}{\partial L} = F_2 A, \quad \frac{\partial Y}{\partial A} = F_2 L. \quad (78)$$

Proposition 5.1 (Growth-accounting equation). *Define the **output elasticities***

$$\alpha_K(t) \equiv \frac{K}{Y} \frac{\partial Y}{\partial K}, \quad \alpha_L(t) \equiv \frac{L}{Y} \frac{\partial Y}{\partial L}.$$

Then

$$g_Y = \alpha_K g_K + \alpha_L g_L + R, \quad R(t) \equiv \frac{A}{Y} \frac{\partial Y}{\partial A} g_A \quad (\text{Solow residual}). \quad (79)$$

Proof. Apply the chain rule to (77) and the product rule to AL :

$$\begin{aligned} Y' &= F_1 K' + F_2 (AL)' = F_1 K' + F_2 (AL' + LA') \\ &= \frac{\partial Y}{\partial K} K' + \frac{\partial Y}{\partial L} L' + \frac{\partial Y}{\partial A} A', \end{aligned}$$

where (78) identifies the three partial derivatives. Divide by Y and insert K/K , L/L and A/A in the respective terms:

$$\begin{aligned} g_Y = \frac{Y'}{Y} &= \underbrace{\frac{K}{Y} \frac{\partial Y}{\partial K}}_{\alpha_K} \underbrace{\frac{K'}{K}}_{g_K} + \underbrace{\frac{L}{Y} \frac{\partial Y}{\partial L}}_{\alpha_L} \underbrace{\frac{L'}{L}}_{g_L} + \underbrace{\frac{A}{Y} \frac{\partial Y}{\partial A} \frac{A'}{A}}_R \\ &= \alpha_K g_K + \alpha_L g_L + R. \end{aligned}$$

□

Corollary 5.2 (What the residual measures).

- (i) $R = \alpha_L g_A$.
(ii) If F has constant returns to scale, then $\alpha_K + \alpha_L = 1$, and (79) can be written per worker:

$$g_{Y/L} = \alpha_K g_{K/L} + R. \quad (80)$$

Proof. (i) Since A and L enter through their product, (78) gives

$$\frac{A}{Y} \frac{\partial Y}{\partial A} = \frac{ALF_2}{Y} = \frac{L}{Y} \frac{\partial Y}{\partial L} = \alpha_L, \quad \text{so} \quad R = \alpha_L g_A.$$

(ii) Constant returns means $F(\lambda K, \lambda AL) = \lambda Y$ for every $\lambda > 0$. Differentiating with respect to λ at $\lambda = 1$ (Euler's theorem) and using (78),

$$K \frac{\partial Y}{\partial K} + L \frac{\partial Y}{\partial L} = Y \implies \alpha_K + \alpha_L = 1.$$

Substitute $\alpha_L = 1 - \alpha_K$ in (79) and subtract g_L from both sides; the growth rate of a ratio is the difference of growth rates, so

$$\underbrace{g_Y - g_L}_{g_{Y/L}} = \alpha_K \underbrace{(g_K - g_L)}_{g_{K/L}} + R. \quad \square$$

The Solow residual is $R = \alpha_L g_A$, not g_A : with $\alpha_L = 2/3$, $R = 1\%$ means $g_A = 1.5\%$. All other terms in (79) are, in principle, observable: g_Y , g_K and g_L from national accounts, and α_K , α_L from income shares (Section 5.2). So R is not measured directly; it is what is left, $R = g_Y - \alpha_K g_K - \alpha_L g_L$. Solow [1957] found that about 7/8 of the growth of U.S. output per hour over 1909–1949 came from R , and only 1/8 from more capital per hour. Abramovitz [1956] called such a residual a **measure of our ignorance**: anything that raises measured output without being counted as an input ends up in R , for example

- better input quality, such as more schooling (Section 5.2, item 2);
- utilization: in a recession, idle machines and workers still count as inputs, so R falls;
- errors in measured capital or in the elasticities (Section 5.2, items 1 and 3).

Reading R as technological progress requires ruling these out.

Remark 5.3 (Productivity conventions). With $Y = K^\alpha (AL)^{1-\alpha} = BK^\alpha L^{1-\alpha}$, where A is *labor-augmenting* technology and $B \equiv A^{1-\alpha}$ is *Hicks-neutral* TFP. Then $R = g_B = (1 - \alpha)g_A$, so reported productivity growth depends on the convention. Only Cobb–Douglas allows this: for other $F(K, AL)$, technology cannot be written as a separate multiplier, $Y = B(A)G(K, L)$. Then there is no TFP level B with $g_B = R$, but Corollary 5.2 still gives $R = \alpha_L g_A$.

5.2 Measuring the terms

Computing R from (79) requires four ingredients.

1. **Capital accumulation.** Capital services are usually assumed proportional to the measured stock. The **perpetual inventory method** combines an initial stock $K(0)$, investment from national accounts and depreciation, $K(t+1) = K(t) + I(t) - \delta K(t)$. Iterating,

$$K(t) = (1 - \delta)^t K(0) + \sum_{s=0}^{t-1} (1 - \delta)^{t-1-s} I(s). \quad (81)$$

For $0 < \delta < 1$, initial-stock errors decay geometrically, while repeated investment errors can persist. Overstating the level of K need not overstate g_K .

Overstating capital growth therefore understates $R = g_Y - \alpha_K g_K - \alpha_L g_L$. For example, with $g_Y = 5\%$, $g_L = 0$ and $\alpha_K = 1/2$, capital growth of 4% gives $R = 5\% - \frac{1}{2}(4\%) = 3\%$. Mistakenly recording $g_K = 8\%$ instead gives $R = 5\% - \frac{1}{2}(8\%) = 1\%$. **The error attributes more of the same output growth to capital.**

2. **Quality of inputs.** Hours worked omit differences in schooling, health and other worker characteristics. Jorgenson and Griliches [1967] aggregate labor categories using wage-bill weights. For a constant-returns aggregate $L = G(L_1, \dots, L_J)$ with marginal-product wages $w_j = wG_j$, where $w = \partial Y / \partial L$, differentiation gives

$$g_L = \sum_j \frac{G_j L_j}{G} g_{L_j} = \sum_j \frac{w_j L_j}{\sum_i w_i L_i} g_{L_j}. \quad (82)$$

The second equality uses Euler's theorem, $G = \sum_i G_i L_i$, so better-paid workers weigh more than in raw hours. The same applies to capital types; omitted quality growth enters R .

3. **Output elasticities.** If capital and labor earn their marginal products, $r = \partial Y / \partial K$ and $w = \partial Y / \partial L$, substituting into the definitions gives $\alpha_K = rK/Y \equiv s_K$ and $\alpha_L = wL/Y \equiv s_L$, so **income shares equal output elasticities** under marginal-product pricing, as in perfect competition. With constant returns, payments exhaust output: $s_K + s_L = 1$ and $Y = rK + wL$.
4. **Annual data.** With $\Delta \log X_t \equiv \log X_t - \log X_{t-1}$, the empirical counterpart is

$$\hat{R}_t = \Delta \log Y_t - \bar{s}_{K,t} \Delta \log K_t - \bar{s}_{L,t} \Delta \log L_t, \quad \bar{s}_{j,t} \equiv \frac{s_{j,t-1} + s_{j,t}}{2}. \quad (83)$$

This is exact for Cobb–Douglas with constant shares. Each weighted term is an input *contribution*; unmeasured quality, utilization and measurement error enter $\hat{R}_t$.

5.3 Why not estimate the elasticities?

Instead of using income shares, could we estimate the elasticities from the relationship between output growth and input growth? Even with perfectly measured inputs, the difficulty is that **productivity affects both output and input choices**. Under constant returns and a constant α_K , (80) is $g_{Y/L} = \alpha_K g_{K/L} + R$. Suppose we regress output growth per worker on capital growth per worker, including an intercept but leaving productivity R unobserved. OLS measures how strongly the two observed growth rates move together. Under standard large-sample conditions, its slope converges to

$$\begin{aligned} \text{plim } \hat{\alpha}_K &= \frac{\text{Cov}(g_{K/L}, g_{Y/L})}{\text{Var}(g_{K/L})} \\ &= \frac{\alpha_K \text{Var}(g_{K/L}) + \text{Cov}(g_{K/L}, R)}{\text{Var}(g_{K/L})} \\ &= \underbrace{\alpha_K}_{\text{true elasticity}} + \underbrace{\frac{\text{Cov}(g_{K/L}, R)}{\text{Var}(g_{K/L})}}_{\text{endogeneity bias}}. \end{aligned}$$

Why the covariance is not zero? Capital is not assigned at random: it responds to productivity. In the Solow model, $K' = sY - \delta K$ by (53), so $g_K = sY/K - \delta$. A period of high productivity growth R raises Y for a given K , hence investment sY and capital growth; with L given, $g_{K/L}$ rises too. Profit-maximizing firms react in the same direction: higher productivity raises the marginal product of capital, so they want more capital. Hence $\text{Cov}(g_{K/L}, R) > 0$, and **OLS attributes part of productivity's effect to capital**, overstating α_K . More data do not remove this bias. Identification requires capital variation unrelated to R , for example through a valid instrument.

5.4 Evidence: primal accounts across countries

Barro and Sala-i Martin [2004], Table 10.1] collect primal accounts, (79) with shares as elasticities, for total GDP (not per capita). Here and below, **TFP growth means the Solow residual R** (for Cobb–Douglas, $R = g_B$). The main patterns:

- **OECD, 1947–73:** TFP accounts for a large part of growth, between a third and two thirds.
- **OECD, 1960–95:** TFP growth is lower in every country appearing in both OECD panels, and capital becomes the main source of growth in most of them.

- **Latin America, 1940–90:** TFP growth is small and capital is the main source almost everywhere. Peru is the only case with negative TFP growth, and its fifty-year average hides sharp episodes such as the 1988–1990 collapse.³⁴
- **East Asia, 1966–90** [Young, 1995]: very fast growth came mostly from input growth. TFP explains a minority of it, and almost none in Singapore.

Two caveats: the OECD panels come from different studies, and the capital shares used for Latin America are high, which by itself raises capital's contribution.

Country	Period	GDP	Capital	Labor	TFP
United States	1947–73	4.0	43	24	34
Japan	1947–73	9.5	35	23	42
United States	1960–95	3.2	37	40	24
Japan	1960–95	5.7	31	22	47
Peru	1940–90	3.2	78	41	−19
Hong Kong	1966–90	7.3	41	28	32
Singapore	1966–90	8.7	65	33	2
South Korea	1966–90	10.3	40	44	16
Taiwan	1966–90	9.4	34	39	28

Table 1: Annual GDP growth (%) and its decomposition (% of growth).

For example, Singapore's 65% is a *share of GDP growth*: capital contributes $0.65 \times 8.7 \approx 5.7$ percentage points a year. Its capital income share is about 0.49. **Young's results** surprised many. Some concluded that East Asian growth was less of a “miracle”: by Lesson 1 of the Solow model (Section 3.2), accumulation alone runs into diminishing returns.

5.5 Dual growth accounting

Hsieh [2002] measured TFP growth from factor *prices* instead of factor *quantities*.

Proposition 5.4 (Primal and dual). *Suppose $Y = rK + wL$, where w is the real wage and r the real rental price of capital. With income shares $s_K = rK/Y$ and $s_L = wL/Y$,*

$$\underbrace{g_Y - s_K g_K - s_L g_L}_{\text{primal}} = \underbrace{s_K g_r + s_L g_w}_{\text{dual}}. \quad (84)$$

Proof. Differentiate $Y = rK + wL$, divide by Y , and rearrange:

$$g_Y = \frac{r'K + rK' + w'L + wL'}{Y} = s_K(g_r + g_K) + s_L(g_w + g_L). \quad \square$$

³⁴For that episode, BCRP real-GDP series PM04901AA gives $Y_{1987} = 200,778$ and $Y_{1990} = 151,492$ in millions of 2007 soles: $100(Y_{1990}/Y_{1987} - 1) \simeq -24.55\%$. Adding the rounded annual rates -9.4% , -12.3% , and -5.0% gives -26.7% , but does not give the compounded change.

The rental price r is the annual cost of using one unit of capital. In the simplest case, $r = i + \delta$: interest on the funds tied up in the machine plus depreciation.

Hsieh's argument. The primal uses capital stocks inferred from investment; the dual uses returns and wages. Hsieh compares them to test the capital data.

1. Solving $s_K = rK/Y$ for r and taking growth rates,

$$r = \frac{s_K}{K/Y}, \quad g_r = g_{s_K} - g_{K/Y}. \quad (85)$$

2. Factor shares were roughly constant, $g_{s_K} \approx 0$, and East Asian national accounts report sharply rising K/Y . Then $g_r < 0$: **if K/Y really rose, the return on capital must have fallen**, since the same share of output pays for more capital.
3. In Korea returns fell, consistent with the capital data, and primal and dual are close (Table 2). In Singapore they did not fall, and the dual far exceeds the primal.
4. Hsieh concludes that Singapore's national accounts overstate investment: overstated g_K depresses the primal residual (Section 5.2, item 1), while the dual does not use K .

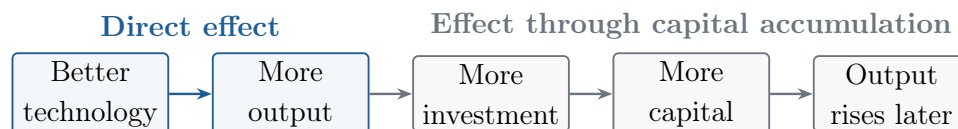
	Hong Kong 1966–91	Singapore 1972–90	South Korea 1966–90	Taiwan 1966–90
Primal	2.3%	−0.7%	1.7%	2.1%
Dual	2.7%	2.2%	1.5%	3.7%

Table 2: Annual TFP growth. Source: **Barro and Sala-i Martin [2004]**, Table 10.2].

Who is right? The gap alone cannot establish that investment was overstated. [Fernald and Neiman \[2011\]](#) find low technology growth in Singapore and show that sectoral distortions can bias both measures. Hsieh's returns come from the unfavored sector and miss falling pure profits: with $Y = rK + wL + \Pi$, primal = dual + Π'/Y .

5.6 Accounting is not causation

Growth accounting attributes growth to inputs, but input growth may itself be caused by technology. A Cobb–Douglas example shows how much this matters.



With a fixed saving rate, higher output raises investment, $I = sY$. Accounting records the resulting capital accumulation as a capital contribution, although technology started the chain.

Let $Y = K^\alpha (Le^{xt})^{1-\alpha}$, with L constant and labor-augmenting technology growing at rate x . Taking logs and differentiating with respect to time,

$$g_Y = \alpha g_K + (1 - \alpha)x. \quad (86)$$

Substituting $g_K = g_Y + g_{K/Y}$ and solving for g_Y ,

$$g_Y = x + \frac{\alpha}{1 - \alpha} g_{K/Y}. \quad (87)$$

What accounting reports. In the Solow and Ramsey models K/Y is constant on the balanced growth path, so (87) gives $g_Y = g_K = x$, and (86) splits growth as

$$\underbrace{x}_{g_Y} = \underbrace{\alpha x}_{\text{capital}} + \underbrace{(1 - \alpha)x}_{\text{measured TFP, } \hat{g}}.$$

But the capital term αx is itself caused by technology.

Proposition 5.5 (Total contribution of technology). *If K/Y is constant, technology accounts for all of output growth, and its total contribution, direct plus through capital, is*

$$g_Y = \frac{\hat{g}}{1 - \alpha} = x. \quad (88)$$

Proof. Constant K/Y means that capital grows because output grows: $g_K = g_Y$. Substituting into (86), with $\hat{g} = (1 - \alpha)x$,

$$g_Y = \alpha g_Y + \hat{g} \implies g_Y = \frac{\hat{g}}{1 - \alpha} = x.$$

If $x = 0$, then $g_Y = 0$: without technology, even capital's share αx disappears. □

The factor $1/(1 - \alpha)$ is a multiplier. Technology raises output directly by $\hat{g}$; capital follows output, adding $\alpha\hat{g}$, which adds $\alpha^2\hat{g}$, and so on: $\hat{g}(1 + \alpha + \alpha^2 + \dots) = \hat{g}/(1 - \alpha)$. This is a **model-based long-run attribution**. With population growth, it applies to output per worker. Extending it to human capital requires that human capital also responds to technology; α is then the elasticity of all reproducible inputs (**broad capital**).

Evidence. Barro and Sala-i Martin [2004, Table 10.3] divide Hsieh's dual TFP growth by $1 - \alpha$, with α each country's capital share (physical capital) or $\alpha = 0.7$ (broad capital).

	GDP growth	TFP, dual	Adjusted, physical	Adjusted, broad
Hong Kong	7.3%	2.7% (37%)	4.3% (59%)	9.0% (123%)
Singapore	8.7%	2.2% (25%)	4.3% (49%)	7.3% (84%)
South Korea	10.3%	1.5% (14%)	2.1% (20%)	5.0% (49%)
Taiwan	9.4%	3.7% (39%)	5.0% (53%)	12.3% (131%)

Table 3: TFP growth adjusted for induced capital; in parentheses, share of GDP growth. Source: **Barro and Sala-i Martin [2004]**, Table 10.3].

Unadjusted, TFP explains a minority of growth; adjusting for physical capital raises this to about half (except in Korea), and for broad capital to over 100% in Hong Kong and Taiwan, a sign that the result depends heavily on α . **A small residual does not mean that technology played a small role.**

5.7 An alternative decomposition

Capital deepening means an increase in K/L : more capital per worker or hour worked. **Here we use K/Y instead:** capital per unit of output. This ratio stays constant when capital and output grow at the same rate, even if capital per worker rises.

Derivation. Let $Y_t = B_t K_t^\alpha H_t^{1-\alpha}$, where H_t is human-capital-adjusted labor and B_t is Hicks-neutral TFP.

1. Divide both sides by Y_t^α :

$$Y_t^{1-\alpha} = B_t \left(\frac{K_t}{Y_t} \right)^\alpha H_t^{1-\alpha}.$$

2. Raise both sides to the power $1/(1-\alpha)$ and define $Z_t \equiv B_t^{1/(1-\alpha)}$:

$$Y_t = \left(\frac{K_t}{Y_t} \right)^{\alpha/(1-\alpha)} H_t Z_t.$$

3. Divide by L_t (workers or hours), take logs and differentiate with respect to time.

$$\frac{Y_t}{L_t} = \left(\frac{K_t}{Y_t} \right)^{\alpha/(1-\alpha)} \frac{H_t}{L_t} Z_t, \quad (89)$$

$$g_{Y/L} = \frac{\alpha}{1-\alpha} g_{K/Y} + g_{H/L} + g_Z, \quad g_Z = \frac{g_B}{1-\alpha}. \quad (90)$$

- **Capital relative to output, K/Y .**
- **Human capital per hour, H/L .**
- **TFP, Z :** since $Y_t = K_t^\alpha (Z_t H_t)^{1-\alpha}$, Z is labor-augmenting technology, and $g_Z = g_B/(1-\alpha)$ is the correction (88) built in.

The approach was popularized by [Klenow and Rodríguez-Clare \[1997\]](#) and goes back at least to [David \[1977\]](#).

Why K/Y and not K/L ? Suppose technology drives balanced growth of 3% in both capital and output, with labor and its quality fixed. Then

$$g_{K/L} = 3\%, \quad g_{K/Y} = 3\% - 3\% = 0.$$

The usual decomposition attributes $\alpha \times 3\%$ to capital, even though technology induced its accumulation. Using K/Y makes the capital term zero and assigns that induced response to productivity instead. **A zero K/Y contribution does not mean capital did not grow:** it grew as fast as output.

This interpretation relies on balanced growth. During a transition, technology can raise output before capital adjusts, lowering K/Y . The ratio alone therefore does not identify what caused accumulation.

Period	Output per hour	$\frac{\alpha}{1-\alpha}g_{K/Y}$	Labor composition	Labor-augmenting TFP
1948–2013	2.5	0.1	0.3	2.0
1948–1973	3.3	−0.2	0.3	3.2
1973–1990	1.6	0.5	0.3	0.8
1990–1995	1.6	0.2	0.7	0.7
1995–2000	3.0	0.3	0.3	2.3
2000–2007	2.7	0.2	0.3	2.2
2007–2013	1.7	0.1	0.5	1.1

Table 4: U.S. private business sector: annual output-per-hour growth (%) and contributions (percentage points), rounded. Source: [Jones \[2016, Table 3\]](#).

5.8 Must productivity grow exponentially?

[Philippon \[2023\]](#) argues that TFP B grows **additively**, by a roughly constant amount each year, not by a constant percentage:

- **Exponential:** $B_{t+\tau} = B_t(1+g)^\tau$, so $\log B_{t+\tau} \approx \log B_t + g\tau$: *log* TFP is linear in time.
- **Additive:** $B_{t+\tau} = B_t + b\tau$: the *level* is linear in time, and the growth rate b/B_t falls as B_t rises.

He uses U.S. TFP for 1947–2019 [[Fernald, 2012](#)] and TFP for 23 countries over 1890–2019 [[Bergeaud et al., 2016](#)]. Fitting both models on the first half of the U.S. post-war sample and forecasting the second half, the additive model tracks TFP closely, while the exponential model overpredicts it more and more (Figure 18).

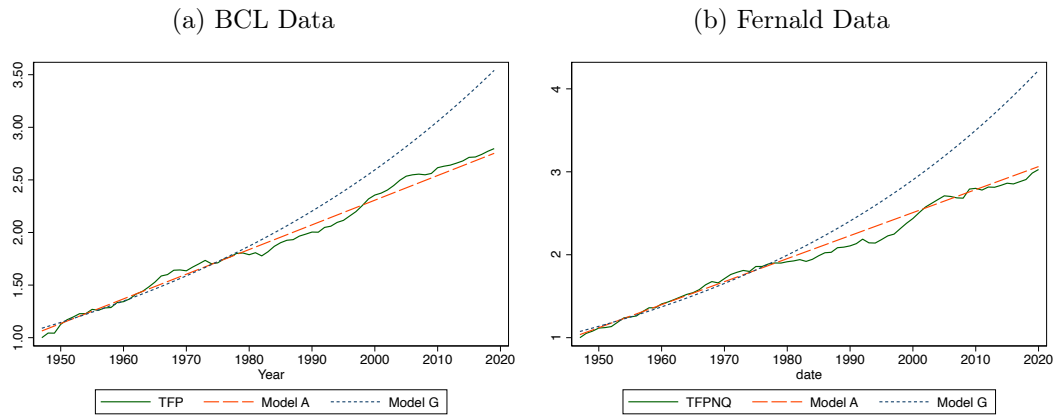


Figure 18: Out-of-sample U.S. TFP forecasts: Model A is additive, Model G exponential; (a) Bergeaud–Cetté–Lecat data, (b) Fernald data. Reproduced from Philippon [2023].

Growth cannot have been linear forever: going back in time, $B_{t-\tau} = B_t - b\tau$ becomes negative for $\tau > B_t/b$. Philippon proposes that general-purpose technologies occasionally raise the slope b : perhaps the Enlightenment or the Glorious Revolution, then the steam engine and electrification. U.S. TFP fits a line with one break around 1930, and U.K. pseudo-TFP³⁵ shows breaks around 1650 and 1830 (Figure 19).

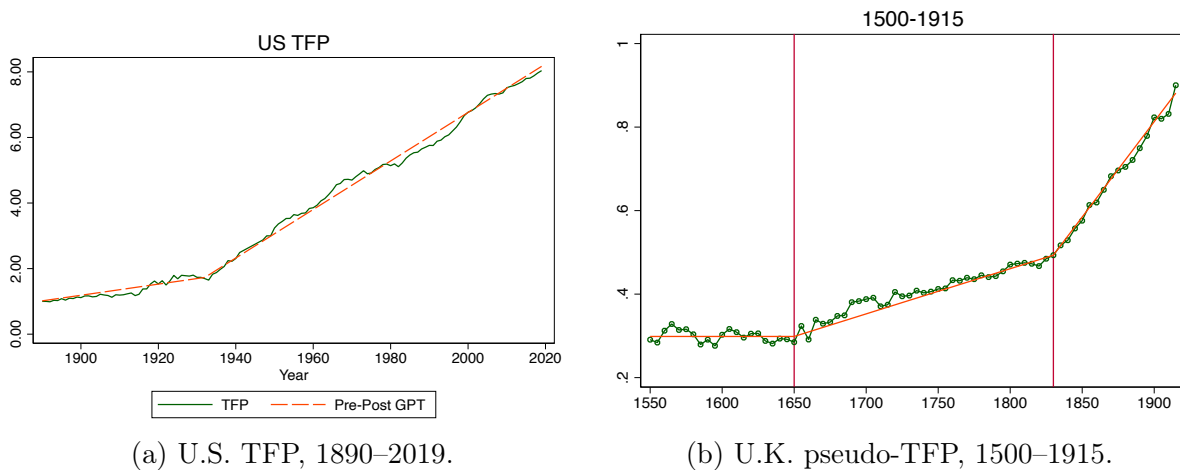


Figure 19: Piecewise-linear TFP fits. Reproduced from Philippon [2023].

³⁵GDP per capita to the power $2/3$. By (89) with constant K/Y and human capital per person, $B \propto (Y/N)^{1-\alpha}$, and $1 - \alpha = 2/3$ for $\alpha = 1/3$.

6 Development Accounting

Growth accounting asks how much of growth over time comes from inputs and how much from productivity. **Development accounting** asks the same question of income differences **across countries at a point in time**: how much of the income gap between rich and poor countries comes from physical capital, human capital and efficiency. As in Section 5, inputs are taken as given; the exercise does not explain why they differ.

Geography and luck, institutions and culture, and policy affect income only through physical capital, human capital and productivity. These three are the **proximate** determinants of income: they enter the production function directly. Development accounting studies only them [Hsieh and Klenow, 2010].

6.1 The framework

Following Hall and Jones [1999], country i produces

$$Y_i = K_i^\alpha (A_i H_i)^{1-\alpha}, \quad H_i = e^{\phi(E_i)} L_i, \quad (91)$$

where Y_i is output, K_i is physical capital, L_i is the number of workers, E_i is average years of schooling, H_i is labor in efficiency units and A_i is labor-augmenting efficiency. The common elasticity $0 < \alpha < 1$ equals capital's income share under competitive factor pricing. Define $y_i \equiv Y_i/L_i$, $k_i \equiv K_i/L_i$ and $h_i \equiv H_i/L_i = e^{\phi(E_i)}$; then $y_i = k_i^\alpha (A_i h_i)^{1-\alpha}$.

Productivity normalization. Below, “TFP” refers to A_i . In the Hicks-neutral form of Section 5.7, $Y_i = B_i K_i^\alpha H_i^{1-\alpha}$, the residual is $B_i = A_i^{1-\alpha}$. Thus $A_i = B_i^{1/(1-\alpha)}$: a percentage difference in A is not the same percentage difference in B .

Repeating the steps of Section 5.7, divide by Y_i^α and raise to $1/(1-\alpha)$:

$$Y_i^{1-\alpha} = \left(\frac{K_i}{Y_i}\right)^\alpha (A_i H_i)^{1-\alpha} \implies Y_i = \left(\frac{K_i}{Y_i}\right)^{\alpha/(1-\alpha)} A_i H_i.$$

Dividing by L_i and taking logs gives

$$\begin{aligned} y_i &= \left(\frac{K_i}{Y_i}\right)^{\alpha/(1-\alpha)} h_i A_i, \\ \log y_i &= \frac{\alpha}{1-\alpha} \log \frac{K_i}{Y_i} + \log h_i + \log A_i. \end{aligned} \quad (92)$$

Equation (92) is an **identity**: it holds for every country as long as the production function (91) is correct. It does not require the economy to be in steady state. The Solow steady state gives the same form with $K/Y = s/(n + g + \delta)$, but here K/Y is the observed

ratio. Expressed as ratios to the United States, the three terms split the income gap into capital intensity K/Y , measured human capital h and a residual A .

Why use K/Y ? The equivalent decomposition in capital per worker is

$$\log y_i = \alpha \log k_i + (1 - \alpha) \log h_i + (1 - \alpha) \log A_i.$$

Capital per worker can respond to productivity and skills. In a model where K/Y stays fixed as A rises, (92) attributes the associated capital accumulation to productivity. This interpretation requires that economic mechanism; the identity alone does not establish it. In particular,

$$\frac{K_i}{L_i} = \frac{K_i}{Y_i} \frac{Y_i}{L_i} :$$

at equal K/Y , a country with one tenth of output per worker also has one tenth of capital per worker. A small contribution from K/Y therefore does not make capital unimportant or imply that investment policies have no effect.

What is in the residual? Rearranging gives $A_i = y_i / [(K_i/Y_i)^{\alpha/(1-\alpha)} h_i]$. It can capture technology, organization, input allocation, omitted school quality and health, measurement error, and misspecification of the production function. It does not identify a single mechanism called technology.

Ratios or logs? Factors of income multiply; log differences add. Define $\Delta_i \log x \equiv \log x_{US} - \log x_i$. For a nonzero income gap, the bilateral log shares are

$$s_{K,i} = \frac{\frac{\alpha}{1-\alpha} \Delta_i \log(K/Y)}{\Delta_i \log y}, \quad s_{h,i} = \frac{\Delta_i \log h}{\Delta_i \log y}, \quad s_{A,i} = \frac{\Delta_i \log A}{\Delta_i \log y}, \quad (93)$$

$$s_{K,i} + s_{h,i} + s_{A,i} = 1.$$

Shares may be negative or exceed one when components offset each other. For example, a country poorer than the United States but with higher K/Y has a negative capital contribution to its income shortfall. These bilateral shares are neither a decomposition of cross-country variance nor causal policy effects. Example 6.1 contrasts them with a statistic based on adding multiplicative factors.

6.2 Measurement

Output. GDP per worker from the Penn World Table [Feenstra et al., 2015], at **purchasing power parity** (PPP). Market exchange rates understate income in poor countries, where many goods, especially services, are cheaper. PPP values every country's output at the same international prices. It does across countries what deflating by a price index does over time.

Physical capital. Perpetual inventory, as in Section 5.2, $K_{t+1} = I_t + (1 - \delta)K_t$. The stock has to start somewhere. We take $K_0 = I_0/(g + \delta)$, with g the earlier growth rate of investment and $\delta = 0.06$ in the early construction. This assumes balanced growth before the sample. The weight of K_0 after t periods is $(1 - \delta)^t$, which is why results are not very sensitive to it [Caselli, 2005].

Human capital. Mincerian wage regressions inform ϕ . Under competitive pay per efficiency unit, a log wage difference associated with schooling can reveal a difference in $\log h$. Interpreting that association as the causal return to schooling requires additional assumptions about selection and other determinants of wages. The standard calibration uses declining slopes, applied to average schooling from Barro and Lee [2013], and normalizes $\phi(0) = 0$:

$$\phi(E) = \begin{cases} 0.134E, & 0 \leq E \leq 4, \\ 0.134 \times 4 + 0.101(E - 4), & 4 < E \leq 8, \\ 0.134 \times 4 + 0.101 \times 4 + 0.068(E - 8), & E > 8. \end{cases} \quad (94)$$

The slopes come from Psacharopoulos [1994]; each applies only to the additional years in its bracket. Schooling years omit quality, health, experience and other skills.

The capital share. The standard choice is $\alpha = 1/3$. Gollin [2002] argues it is reasonable once self-employment income is split between capital and labor; Feenstra et al. [2015] disagree. Caselli [2005] finds that results are sensitive to α , and much less to depreciation, the initial capital stock, schooling returns and measurement, and hours worked.

- **A large share of the gap is residual.** Roughly half or more of income differences are attributed to productivity. The exact share depends on the comparison, the definition of a share and how inputs are measured.
- **Schooling accounts for a substantial part**, perhaps a third of the gap between rich and poor countries. Migration evidence suggests that broader human capital accounts for more (Section 6.4).
- **The capital–output ratio accounts for little:** about $\log 1.8 / \log 31.7 \approx 17\%$ of the log gap between the five richest and the five poorest countries in 1988 (Table 5), and less in 2010. Similar K/Y is consistent with capital adjusting to productivity and skills, even when K/L differs greatly.
- **Capital measurement may explain part of this change.** Structures depreciate slowly, so they weigh more in capital than in investment, and are cheap in poor countries. Newer series account for this, raising measured capital in poor countries.

- **The components suggest where to look for frictions.** A low K/Y points to capital taxes, financial repression or barriers to inflows; a low h points to access to schooling or demand for it. The residual A contains omitted school quality.
- **This reversed the earlier consensus.** [Mankiw et al. \[1992\]](#) attributed most cross-country variation to physical and human capital. [Klenow and Rodríguez-Clare \[1997\]](#) questioned their schooling measure and their human-capital technology, and [Hall and Jones \[1999\]](#) measured human capital with Mincerian returns and used the K/Y decomposition.

	Y/L	$(K/Y)^{\alpha/(1-\alpha)}$	H/L	A
United States	1.000	1.000	1.000	1.000
Japan	0.587	1.119	0.797	0.658
Mexico	0.433	0.868	0.538	0.926
India	0.086	0.709	0.454	0.267
China	0.060	0.891	0.632	0.106
Kenya	0.056	0.747	0.457	0.165
Average, 127 countries	0.296	0.853	0.565	0.516
Correlation with Y/L , logs	1.000	0.624	0.798	0.889
Five richest / five poorest	31.7	1.8	2.2	8.3

Table 5: Development accounting in 1988, ratios to the U.S., [Hall and Jones \[1999, Table I\]](#).

	y	$(K/Y)^{\alpha/(1-\alpha)}$	h	TFP	Share due to TFP
United States	1.000	1.000	1.000	1.000	—
Japan	0.683	1.218	0.903	0.620	64%
Mexico	0.338	0.931	0.760	0.477	60%
China	0.136	1.137	0.713	0.168	83%
Kenya	0.037	0.819	0.618	0.073	87%
Malawi	0.021	1.107	0.507	0.038	94%
Average, 128 countries	0.212	0.979	0.705	0.307	64%
1/Average	4.72	1.02	1.42	3.26	69%

Table 6: Same for 2010, [Jones \[2016, Table 6\]](#). Last column: Example 6.1.

In both tables the capital column is near one for rich and poor countries, the schooling column falls with income but much less than output per worker, and the productivity column tracks income most closely (see the correlations in Table 5). The two tables differ in year, sample and data construction, so their difference alone does not show a trend from 1988 to 2010.

Example 6.1 (Mexico against the United States, 2010). Express y, h, A relative to the United States and let $c = [(K/Y)_{\text{MEX}}/(K/Y)_{\text{US}}]^{\alpha/(1-\alpha)}$. **Calculate productivity**

as the residual, then its share of the log gap: $A = \frac{y}{c \cdot h} = \frac{0.338}{0.931 \cdot 0.760} \approx 0.48$ and $s_A = \frac{\log(1/A)}{\log(1/y)} \approx \frac{2}{3}$. The factors multiply, but their log contributions add: $\log(1/y) = \log(1/c) + \log(1/h) + \log(1/A)$. Compute the other shares by replacing A with c or h in s_A . Productivity accounts for most of this gap under the measured inputs; this does not identify its cause. The table instead reports $(1/A)/[1/(c \cdot h) + 1/A]$, which adds multiplicative factors and is not a log share. If given schooling rather than h , use $h = e^{\phi(E_{\text{MEX}}) - \phi(E_{\text{US}})}$; within one return bracket this simplifies to $e^{r(E_{\text{MEX}} - E_{\text{US}})}$. Here $c \cdot h$ is the product of the physical-capital contribution c and human capital per worker h , both measured relative to the United States.

6.3 Why doesn't capital flow to poor countries?

Little capital per worker does not necessarily mean a high return to investment. The **Lucas puzzle** starts from diminishing returns: an extra machine should produce more where machines are scarce, attracting investment to poor countries. But this prediction holds productivity and worker skills equal. A machine may contribute little if the workers and production process using it are also less productive.

What determines the return? With $y = \tilde{B}k^\alpha$, where $\tilde{B}$ includes productivity and worker skills, $\text{MPK} = \alpha \tilde{B}k^{\alpha-1}$ is the extra output from another unit of capital. Using the production function,

$$\text{MPK} = \alpha \frac{Y}{K} = \frac{\alpha}{K/Y}. \quad (95)$$

So compare **capital relative to output**, not just capital per worker. With the same α , half the K/Y means twice the MPK. But if capital and output are both proportionally lower, K/Y and MPK are unchanged. The similar K/Y ratios in Tables 5 and 6 therefore weaken the prediction of enormous returns in poor countries [Lucas, 1990].

Two measurement corrections [Caselli and Feyrer, 2007].

- **Match income to the capital stock.** If K measures machines and buildings, exclude income earned by land and natural resources. A tractor should not receive credit for all the income of a farm. Use the income share of reproducible capital, α_K , rather than all nonlabor income.
- **Compare revenue with the cost of investing.** A machine may cost about the same across countries while its output sells for less in a poor country. The same physical MPK then yields less revenue per unit invested. Let P_y be the output price and P_k the capital price.

The **rental return**, r_{rental} , is annual income earned by capital per unit of its purchase value, before depreciation. “Rental” refers to capital’s services, even when the firm owns the machine.

1. **Physical productivity.** One extra unit of capital produces MPK additional units of output per year.
2. **Convert output into money.** Each unit of output sells for P_y , so the extra capital generates $P_y \cdot \text{MPK}$ in annual income. Divide by P_k , the cost of buying that unit of capital, to get its return.
3. **Use the correct capital share.** Under competitive factor pricing, total income earned by reproducible capital satisfies $P_y \cdot \text{MPK} \cdot K = \alpha_K \cdot P_y \cdot Y$. Cancelling P_y and dividing by K gives $\text{MPK} = \alpha_K Y / K$.

$$\begin{aligned}
 r_{\text{rental}} &= \frac{\text{annual income from one unit of capital}}{\text{purchase price of that unit}} \\
 &= \frac{P_y \cdot \text{MPK}}{P_k} = \frac{P_y}{P_k} \cdot \frac{\alpha_K Y}{K} = \frac{\alpha_K \cdot P_y Y}{P_k K}.
 \end{aligned} \tag{96}$$

The last fraction uses the whole stock: $\alpha_K P_y Y$ is **income earned by machines and buildings**; $P_k K$ is **their value**. It is capital income divided by capital value.

Example. With $P_k = 100$, $\text{MPK} = 5$ units per year and $P_y = 2$, the return is $(2 \cdot 5)/100 = 10\%$. At $P_y = 1$, the same machine costs the same and produces the same, but its return falls to 5%.

Allowing for wear. With annual depreciation δ and unchanged capital prices, the net return is $r_{\text{rental}} - \delta$; the gross return $1 + r_{\text{rental}} - \delta$ includes income and the remaining machine value. Investors compare these financial returns at comparable risk.

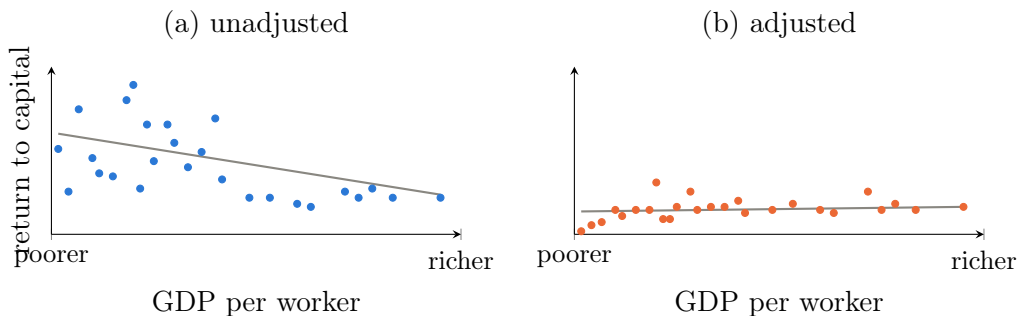


Figure 20: Returns in 1996, schematic. Moving right means richer countries; moving up means higher measured returns. (a) Raw estimates suggest higher returns in poor countries. (b) Correcting capital income and prices largely removes this pattern. Both panels refer to the same year.

6.4 Human capital: evidence from migration

Hendricks and Schoellman [2018] compare the **same worker's wages before and after migration** to separate worker skills from country conditions. Write output per worker as

$$y_c = z_c h_c, \quad z_c = \left(\frac{K_c}{Y_c} \right)^{\alpha/(1-\alpha)} A_c. \quad (97)$$

Here h_c is average human capital; z_c combines capital intensity and productivity. With competitive pay per efficiency unit, common α and fully transferable skills h_i , wages satisfy

$$w_{i,c} = (1 - \alpha) z_c h_i \implies \frac{w_{i,US}}{w_{i,c}} = \frac{z_{US}}{z_c}. \quad (98)$$

The worker's skills cancel, so the wage gain measures the country effect. Compare it with the countries' income gap, using logs to obtain shares:

$$s_{\text{country}} = \frac{\log(w_{i,US}/w_{i,c})}{\log(y_{US}/y_c)}, \quad s_h = 1 - s_{\text{country}}. \quad (99)$$

The country share includes physical capital, not just TFP.

Origin, GDP per worker relative to U.S.	Wage gain	GDP gap	h share	95% CI
below 1/16	3.2	31.8	0.66	(0.60, 0.73)
1/16 to 1/8	2.8	11.9	0.58	(0.54, 0.62)
1/8 to 1/4	1.9	5.6	0.63	(0.55, 0.71)
1/4 to 1/2	1.8	3.0	0.48	(0.34, 0.62)
1/2 to 1	1.2	1.3	0.48	(−0.23, 1.19)
Mexican migrants, MP	2.0	2.9	0.33	(0.29, 0.37)

Table 7: Hendricks and Schoellman [2018, Table II]. Wage gain: U.S./origin hourly pay of the same workers; GDP gap: U.S./origin output per worker. Ratios are rounded; shares are as reported. First five rows: New Immigrant Survey; MP: Mexican Migration Project.

- **Country conditions matter:** the same workers earn more after migrating, even though their skills are held fixed in the comparison.
- **Skills explain much of the income gap:** wage gains are much smaller than the GDP gaps, implying a large human-capital share. Human capital matters more than schooling years alone suggest; some of the productivity residual may be omitted skills.

Identification: comparing the same worker removes fixed skill differences, but time effects and selection in migration gains can remain. If only τh_i transfers, then $\Delta \log w_i = \Delta \log z + \log \tau$. For $\tau < 1$, language or credential barriers depress the wage gain, so ignoring them **understates the country share and overstates the human-capital share**.

6.5 Agriculture

Subscripts a and n mean agriculture and non-agriculture. L_a and L_n count workers in each sector. **VA is one abbreviation: value added**, meaning output value minus the cost of purchased inputs. Thus VA_a/L_a is value added per farm worker. The agricultural productivity gap (APG) is

$$\text{APG} \equiv \frac{VA_n/L_n}{VA_a/L_a} = \frac{\text{value added per non-farm worker}}{\text{value added per farm worker}}. \quad (100)$$

$\text{APG} > 1$ means higher value added per worker outside farming.

Equal-share Cobb–Douglas benchmark. With no purchased intermediates, let both sectors have the same labor elasticity $0 < \vartheta < 1$:

$$Y_s = A_s K_s^{1-\vartheta} L_s^\vartheta, \quad s \in \{a, n\}.$$

Normalize the non-farm price to one and denote the farm price by p_a . Identical workers, equal hours and free movement imply a common competitive wage w . The labor first-order conditions give

$$\begin{aligned} w &= p_a \frac{\partial Y_a}{\partial L_a} = \vartheta \frac{p_a Y_a}{L_a}, \\ w &= \frac{\partial Y_n}{\partial L_n} = \vartheta \frac{Y_n}{L_n}, \\ \text{APG} &= \frac{Y_n/L_n}{p_a Y_a/L_a} = 1. \end{aligned} \quad (101)$$

This equates **value added per worker**, not physical output: here $VA_a = p_a Y_a$ and $VA_n = Y_n$.³⁶

Production and barriers. The model of Restuccia et al. [2008] adds land, inputs and wage barriers. Y_a, Y_n are output quantities; $N = L_a + L_n$ is total workers:

$$Y_a = X^\alpha \left[Z^{1-\sigma} (\kappa A L_a)^\sigma \right]^{1-\alpha}, \quad Y_n = A L_n. \quad (102)$$

X is farm inputs (e.g. fertilizer); Z is fixed land. A is productivity in both sectors; κ is a farm productivity factor. α measures output's response to inputs; σ weights labor relative to land ($0 < \alpha, \sigma < 1$). p_a is the food price; other goods have price one. w_a, w_n are wages: competition gives $w_n = A$. The input price $\pi > 1$ is the **direct barrier**; the wage reduction $\theta \in [0, 1)$ is the **indirect barrier**, with $w_a = (1 - \theta)A$.

³⁶Neither equal sectoral productivity nor capital mobility is needed. Different A_s and sector-specific K_s affect labor allocation, but common wages and equal labor shares still equate value added per worker.

Farm choices. A farm buys inputs and hires workers until the extra revenue from each equals its extra cost:

$$p_a \alpha \frac{Y_a}{X} = \pi \implies x \equiv \frac{X}{Y_a} = \alpha \frac{p_a}{\pi}, \quad (103)$$

$$p_a \sigma (1 - \alpha) \frac{Y_a}{L_a} = w_a = (1 - \theta)A. \quad (104)$$

Here $x = X/Y_a$ is inputs per unit of farm output; p_a/π is the food price relative to the input price.

Food demand. c_a, c_n are consumption per worker; y is income and U utility. $\bar{a}$ is minimum food consumption. The parameter $a \in (0, 1)$ is the food share of spending *after* buying this minimum. With $U = a \log(c_a - \bar{a}) + (1 - a) \log c_n$ and budget $p_a c_a + c_n = y > p_a \bar{a}$, demand is

$$c_a = \bar{a} + \frac{a}{p_a}(y - p_a \bar{a}), \quad c_n = (1 - a)(y - p_a \bar{a}), \quad (105)$$

How many workers must farm? To focus on basic food needs, take $a \rightarrow 0$: each worker consumes exactly $\bar{a}$ units of food. Without food trade, the country must produce $Y_a = N\bar{a}$.³⁷ Let $z = Z/N$ be land per worker and $\ell_a = L_a/N$ the share working in farming. Replace X with xY_a in the production function and divide by N to get $\bar{a} = x^{\alpha/(1-\alpha)} z^{1-\sigma} (\kappa A \ell_a)^\sigma$. Solving for the share of workers in farming:

$$\ell_a = \frac{1}{\kappa A} \left[\frac{\bar{a}}{z^{1-\sigma} x^{\alpha/(1-\alpha)}} \right]^{1/\sigma}, \quad \ell_a = \frac{\bar{a}}{Y_a/L_a}. \quad (106)$$

The second expression says: food needed per worker divided by output per farmer. Combining this result with the two farm choices gives, with other parameters fixed,

$$x \propto \left(\frac{1 - \theta}{\pi} \right)^{\frac{\sigma(1-\alpha)}{\alpha + \sigma(1-\alpha)}}.$$

The exponent is positive, so a higher π or θ reduces input use. Dividing the two farm conditions gives $X/L_a = [\alpha/(\sigma(1-\alpha))](w_a/\pi)$. A lower farm wage makes hiring workers cheaper relative to buying inputs, so farms use fewer inputs per worker. This is how the wage gap creates an indirect barrier.

- **Why do so many people work in low-productivity agriculture?** The model must explain both why output per farmer is low and why so many workers still farm.

³⁷Imports can reduce the need to produce food at home, but trade costs and low export earnings can limit this option [Tombe, 2015].

- **Low input use explains low productivity.** Direct and indirect barriers reduce modern inputs per farmer, X/L_a . With less fertilizer and other inputs, each farmer produces less food.
- **Stone–Geary preferences explain why the output is still needed.** People must eat even when they are poor. In the subsistence limit, food demand is $N\bar{a}$; without food imports, low output per farmer requires a high farm employment share, $\ell_a = \bar{a}/(Y_a/L_a)$.
- **Workers are crowded into agriculture.** More workers share fixed land, reducing land per farmer and reinforcing low farm productivity. Fewer workers produce non-agricultural goods, lowering aggregate income.
- **How much of the income gap does this explain?** Restuccia et al. [2008, Table 2] report a rich-to-poor ratio of **34.3 to 1** in aggregate output per worker. Their model generates 10.8 with both barriers, versus 5.5 without barriers: the mechanism amplifies the gap but explains only part of it. Productivity, barriers and land per worker remain exogenous, so their cross-country differences are not explained.

6.6 Absolute and comparative advantage

A higher average income elsewhere does not mean every worker would earn more by moving.

1. Why might workers stay?

- **Selection:** people in different places have different skills; the income gap need not come from the place itself.
- **Frictions:** moving has costs or barriers, even when earnings would be higher elsewhere.

2. Two kinds of skill differences.

- **Absolute advantage:** one person is better at both fishing and other work. A worker fixed effect captures a common skill component.
- **Comparative advantage:** workers differ in which activity they do relatively well. One worker fixed effect does not capture these differences across activities. This is the **Roy model** [Roy, 1951].

3. The moving decision. Compare the *same worker's* options:

- q : the worker's type. Higher q means a stronger comparative advantage outside fishing.
- $\bar{Y}_F(q)$ and $\bar{Y}_N(q)$: expected log earnings in fishing (F) and non-fishing (N), respectively.

- $m \geq 0$: the cost of moving, in the same units as the graph. Move when $\bar{Y}_N(q) - m > \bar{Y}_F(q)$.

The two cutoffs. $\tilde{q}$ is the type just indifferent without a moving cost; q^* is the type just indifferent with the cost:

$$\bar{Y}_N(\tilde{q}) = \bar{Y}_F(\tilde{q}), \quad \bar{Y}_N(q^*) - m = \bar{Y}_F(q^*). \quad (107)$$

- $q < \tilde{q}$: fishing pays more, so the worker stays.
- $\tilde{q} < q < q^*$: outside earnings are higher, but the gain does not cover m , so the worker stays.
- $q > q^*$: the gain covers m , so the worker moves.

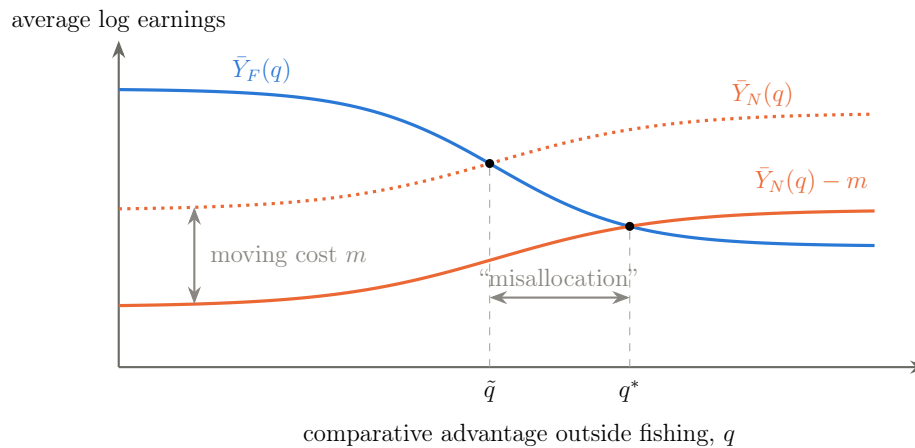


Figure 21: Moving costs and the decision to leave fishing.

How to read the graph.

1. **Axes.** q orders workers by their advantage outside fishing. Height is expected log earnings for that worker type.
2. **Lines.** Blue is fishing. Dotted is outside earnings before costs. The lower solid line subtracts m . The vertical gap is m , in log-earnings units.
3. **Crossings.** At $\tilde{q}$, fishing equals outside earnings before costs; at q^* , after costs. Only workers to the right of q^* move.
4. **Between the cutoffs.** Outside earnings are higher before costs, but the gain cannot cover m , so workers stay. Lowering m raises the solid outside line and encourages moving.

Evidence on moving.

- **Bangladesh:** a randomized travel subsidy raised consumption [Bryan et al., 2014].
- **Finland:** forced migration after WWII raised incomes for people initially in agriculture (Sarvimäki, Uusitalo and Jäntti, 2022).

- **United States:** moving vouchers improved long-run outcomes for children who moved young [Chetty et al., 2016].
- **Iceland:** *The Gift of Moving* [Nakamura et al., 2022] finds that volcano-induced moves raised children’s earnings and education. Leaving a rich fishing town can improve the match between skills and jobs, while parents may bear the costs.

What to conclude. These studies show gains for particular groups, not a common gain for everyone. The graph’s “misallocation” means lower earnings before moving costs. If m is a real cost, higher earnings alone do not establish that moving makes the worker better off.

7 Fundamental Causes of Growth

Development accounting describes income differences through physical capital, human capital, and productivity. The next question is why these inputs and their use differ across countries: the **fundamental causes** of growth. Colonial institutions, national leaders, and political obstacles to reform offer three connected ways to investigate this question.

7.1 Causes of growth and the identification problem

Figure 22 arranges the candidate causes as a chain [Hsieh and Klenow, 2010]. Fundamental causes shape income through their effects on factor accumulation and productivity.

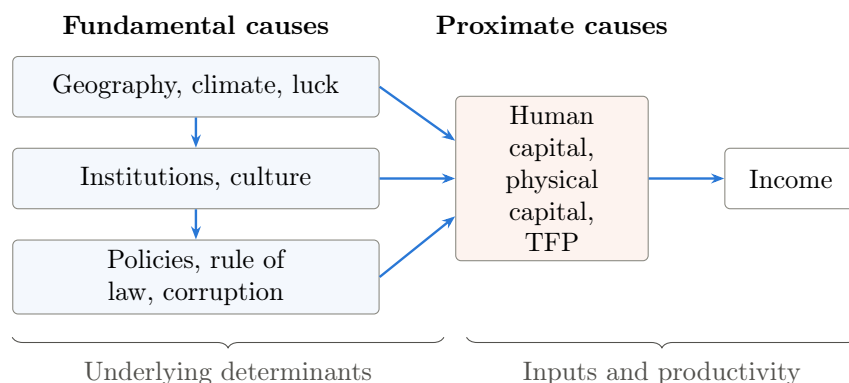


Figure 22: Fundamental and proximate causes of income differences, after Hsieh and Klenow [2010]. Each fundamental cause can shape the one below it and acts on income through human capital, physical capital, and TFP.

- Geography may influence institutions, which then shape policies, investment, and productivity. Trade and religion are further candidate causes.
- Finding that capital accounts for an income gap does not establish why the gap arose. It identifies a channel through which deeper causes operate.

The **institutional hypothesis** holds that good institutions cause high income. It concerns the incentives that rules create, not just whether a particular government currently chooses good policies:

- **Secure property rights and rule of law.** An investor can expect to retain more of an investment's return.
- **Constraints on the executive.** That expectation stays credible even when the identity or preferences of the ruler change.

Before the late 1990s this literature was **mostly qualitative**. North and Thomas [1973], Jones [1981], North and Weingast [1989], North [1990], and Engerman and Sokoloff [1997] built the case through historical analysis, not statistical estimation. A systematic test needs **exogenous variation** that separates the effect of institutions from other differences between countries. The quantitative work of the 1990s, cross-country growth regressions [Barro, 1991; Sala-i Martin, 1997], face:

- **Reverse causality.** Higher income may produce democracy, better rule of law, or greater government spending; this is the modernization hypothesis.
- **Omitted variables.** A third factor may cause both growth and the variable being studied.
- **Level versus growth effects.** Many changes raise the long-run income level without permanently raising its growth rate. Slow transition toward that higher level can nevertheless produce decades of faster growth.

A positive association between institutional quality and income is compatible with several causal stories. The empirical task is to isolate institutional variation that is not itself a response to prosperity or to another determinant of prosperity.

7.2 Colonial origins: institutions and income

Acemoglu et al. [2001] turned the qualitative institutional argument into a statistical design: a historical source of variation in institutions, used as an instrument. The paper was, and remains, **highly controversial**:

- **Critics.** The ideas are not original, the data are of low quality, the identifying restriction is implausible, the handling of the data is questionable, and the standard errors are wrong.
- **Defenders.** Formal data and statistical methods discipline the analysis. Traditional historical analysis can pick anecdotes to fit the story and is hard to replicate. The paper is easy to criticize because its methods are explicit. It pioneered causal historical analysis, and its authors shared the 2024 Nobel Prize.

7.2.1 The mechanism and empirical specification

Acemoglu et al. [2001] argue that:

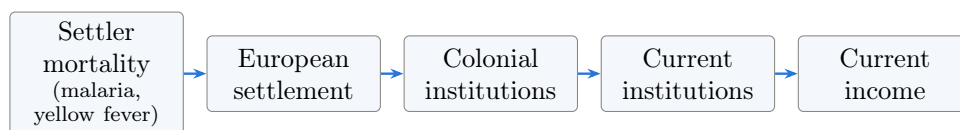


Figure 23: The institutional channel proposed by **Acemoglu et al. [2001]**.

The empirical specification has two stages:

$$\text{Second stage:} \quad \log y_i = \mu + \alpha R_i + X_i' \gamma + \varepsilon_i, \quad (108a)$$

$$\text{First stage:} \quad R_i = \zeta + \beta \log M_i + X_i' \delta + v_i. \quad (108b)$$

Here y_i is GDP per capita in 1995 at purchasing power parity; R_i is Political Risk Services' protection against expropriation, averaged over 1985–1995; M_i is historical mortality; and X_i contains controls. **Higher R_i means greater protection**, despite the abbreviated expropriation-risk label in the plots. The first stage (108b) isolates the variation in protection predicted by mortality; the second stage (108a) relates income to that variation. AJR predicts $\beta < 0$ and $\alpha > 0$: higher mortality discouraged settlement and weakened protection, while stronger protection raised income.

The **reduced form** regresses income directly on mortality:

$$\log y_i = \mu_{\text{RF}} + \lambda \log M_i + X_i' \gamma_{\text{RF}} + u_i.$$

Write $Y_i = \log y_i$ for the outcome and $Z_i = \log M_i$ for the instrument. With one instrument, the same sample, and the same controls,

$$\hat{\alpha}_{\text{IV}} = \frac{\hat{\lambda}}{\hat{\beta}} = \frac{\widehat{\text{Cov}}(Y_i, Z_i) / \widehat{\text{Var}}(Z_i)}{\widehat{\text{Cov}}(R_i, Z_i) / \widehat{\text{Var}}(Z_i)} = \frac{\widehat{\text{Cov}}(Y_i, Z_i)}{\widehat{\text{Cov}}(R_i, Z_i)}, \quad (109)$$

where the sample moments use each variable net of the controls X_i . The ratio converts an income change per unit of log mortality into an income change per unit of protection. Under the institutional channel, $\lambda = \alpha\beta$: a negative first stage and a negative reduced form imply a positive institutional effect.

7.2.2 Relevance, exclusion, and the plots

Relevance means mortality predicts institutions conditional on controls. The **exclusion restriction** requires mortality to affect current income only through institutions; the instrument must also be unrelated to other determinants of income left in the error term. The main threats are:

- **Persistent disease.** Historical mortality may predict disease today, which directly reduces productivity. AJR argues that surviving local adults had much lower mortality than European newcomers³⁸ and that yellow fever was subsequently largely controlled. The counterargument is that low adult mortality does not eliminate illness, childhood effects, or the tsetse fly's effects on livestock [Alsan, 2015].

³⁸AJR attributes this difference to acquired immunity among local adults, in contrast to newcomers without that protection [Acemoglu et al., 2001].

- **Geography, climate, and trade.** The disease environment goes with tropical location and weather, which may lower agricultural productivity directly, and with poor access to trade. Each is a path from mortality to income that bypasses institutions [Sachs, 2003]. Controls for latitude and continent address this only in part.
- **Human capital.** Settlers brought skills as well as institutional arrangements. If those skills independently raised later income, mortality would not isolate the institutional channel [Glaeser et al., 2004].

All three objections concern alternative paths from the instrument to income. They can therefore survive a very strong first stage. In particular, showing that Europeans responded to disease conditions does not tell us whether those same conditions also affected later productivity directly.

The first-stage plot has log mortality on the horizontal axis and protection on the vertical axis; the reduced-form plot replaces protection with log income. Both slopes are negative. They support the proposed relationships, but **relevance does not establish exclusion**.

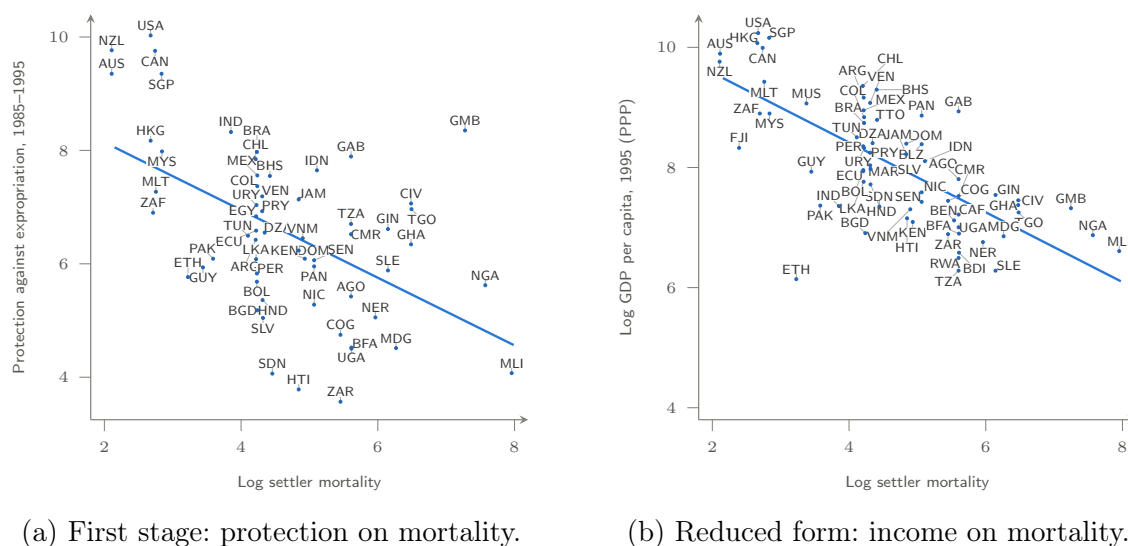


Figure 24: Mortality, protection, and income [Acemoglu et al., 2001]. Reconstructions of the published plots; country coordinates and fitted lines are approximate graphical readings.

7.2.3 Results and interpretation

- **Strong positive relationship.** Acemoglu et al. [2001] find that greater protection against expropriation goes with higher income.
- **IV exceeds OLS.** A larger estimate does not establish that IV is correct: its causal interpretation still depends on the instrument assumptions.

- **Stable across samples and controls.** The relationship survives excluding rich settler economies or Africa and adding controls. It is not driven solely by either group, although its magnitude changes across samples.
- **Robustness does not establish exclusion.** Alternative channels from mortality to income remain possible.
- **Large implied effect.** If the instrument is valid, institutional differences can explain a substantial part of international income gaps.
- **Levels, not growth rates.** The estimates concern the **level of income**. They do not imply permanently faster growth or that reform delivers the gains immediately.

7.3 Mortality data and the AJR–Albouy debate

The instrument relies mainly on **soldier mortality** as a proxy for the risks facing potential settlers. [Albouy \[2012\]](#) questions whether the observations measure those risks comparably:

- **Geography.** Many countries are assigned mortality rates observed elsewhere.
- **Populations.** Latin American rates come from sparse bishop records, rescaled to compare with soldiers.
- **Circumstances.** Campaign and barracks mortality may reflect different military conditions as well as disease risk.

Figure 25 separates observations by source and circumstances. The concern is that the estimated relationship may depend on how unlike observations are assigned and converted into a common measure.

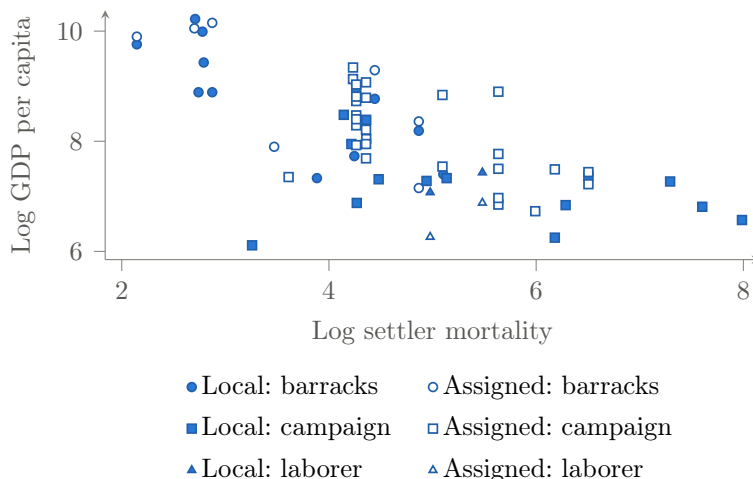


Figure 25: Income and mortality by the source and circumstances of the mortality observation [[Albouy, 2012](#)]. Reconstruction from the coordinates of the published figure.

Acemoglu et al. [2012] reply that results survive limiting extreme mortality rates and changing the bishop-to-soldier conversion. They also argue that Albouy's preferred estimates depend heavily on Gambia and that campaign and barracks conditions were not sharply distinct.

The debate separates **measurement robustness** from **causal validity**. A result that survives reasonable ways of constructing mortality is more credible as an empirical pattern. It still requires the argument that mortality affects income only through institutions. The design's explicit data and assumptions make both questions open to scrutiny.

7.4 Korea: institutions or lucky leaders?

Does growth depend on lasting rules or on who happens to be in charge?

- **Good policies can come before strong rules.** Both Koreas had weak constraints on the executive during much of their early economic divergence. South Korea grew rapidly under Park Chung-hee before strong limits on the leader's power developed [Glaeser et al., 2004].
- **A good leader is different from a lasting rule.** Respect for property may reflect a ruler's choice rather than an enforceable constraint; a successor could reverse that choice. Durable rules make protection less dependent on who governs.
- **What Korea shows.** Strong limits on leaders need not come before growth. This does not prove that institutions are unimportant or that leadership alone explains Korea's success.

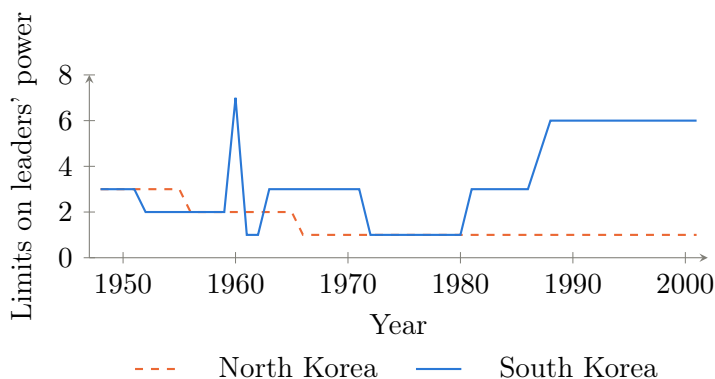


Figure 26: Limits on leaders' power in Korea [Glaeser et al., 2004]. Low values mean few constraints on the executive; South Korea's sustained increase came after its early period of rapid growth.

7.5 Do leaders matter?

7.5.1 Why study leaders' deaths?

Jones and Olken [2005] ask whether changing a country's leader changes its economic growth.

- **The economy also affects who stays in power.** A recession can bring down a leader. Simply comparing leaders could confuse the reason they lost office with the effect of replacing them.
- **Deaths offer a useful comparison.** The study looks at leaders who die from illness or accidents while in office. These deaths are less likely to be caused by economic conditions than ordinary changes of leader.
- **Compare the years before and after.** The study uses equal periods on either side of the death, leaving out the death year. Comparing whole periods in office could be misleading because economic conditions may have helped a leader come to power in the first place.

Figure 27 shows four dramatic cases. Each panel has three lines:

- **Blue line: income per person.** A steeper upward slope means faster growth.
- **Dashed grey line: the leader takes office.** Mao in China (1949), Machel in Mozambique (1975), Touré in Guinea (1958), and Khomeini in Iran (1979).
- **Solid orange line: the leader dies.** Mao in 1976, Machel in 1986, Touré in 1984, and Khomeini in 1989.

In all four countries income grows faster after the death than during the leader's rule. Other events can also change growth, so these pictures alone do not establish the leader's effect.

7.5.2 What does the evidence suggest?

- **Leaders can matter for growth.** The study finds evidence of changes beyond the economy's usual ups and downs, especially for leaders who had been in office longer. Some comparisons give clearer evidence than others.
- **A change can help or hurt.** Losing a capable leader may slow growth; replacing a harmful one may improve it. These effects can cancel in an average even when individual leaders matter.
- **Rules can limit one person's influence.** The evidence is stronger for dictators than for democratic leaders. This fits the idea that laws and other limits on power reduce how much one person can change the economy. It does not mean that democratic leaders have no effect.

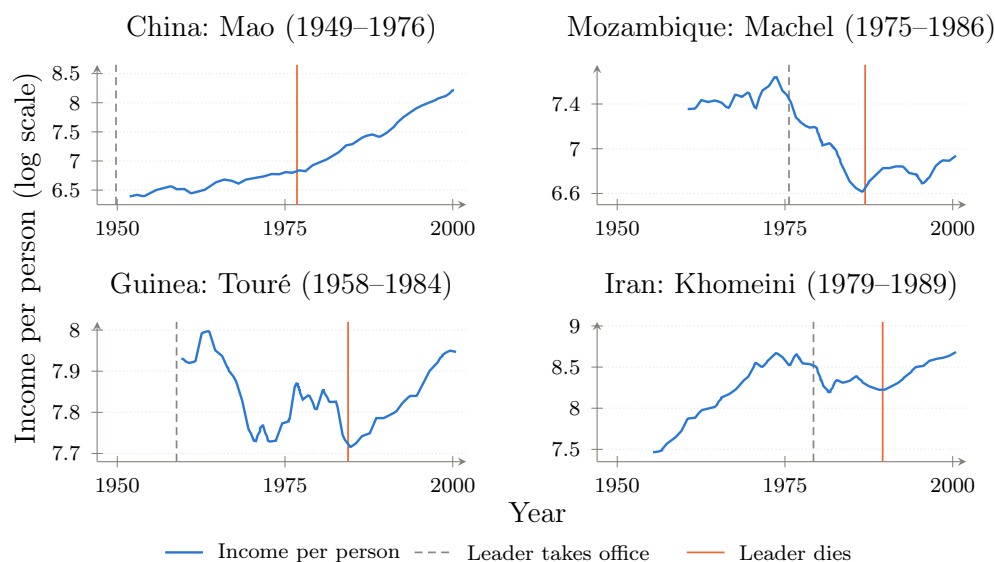


Figure 27: Income before and after four leaders' deaths [Jones and Olken, 2005].

7.6 Why are good institutions not adopted?

Reforms can make a country richer while hurting a powerful group, the elite, that benefits from existing arrangements. The elite may block change. If the rest of the population gains enough, why does it not **compensate** the elite: hand over part of its gains, for example through taxes and transfers, so that the elite also comes out ahead and accepts reform?

The **Coase theorem** suggests that people can negotiate a deal that avoids wasting these gains, provided bargaining is costless and agreements can be enforced. Politics can make these conditions hard to meet.

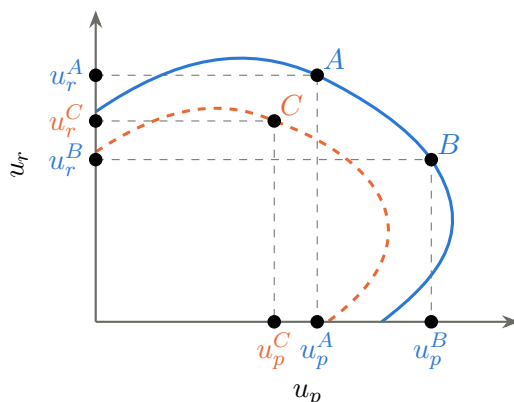


Figure 28: Reform and compensation. C is the starting point. B is reform without compensation: the public gains and the elite loses. A is reform with a transfer from the public to the elite, which benefits both groups.

Figure 28 shows the public's well-being (u_p) horizontally and the elite's well-being (u_r) vertically. Reform moves the country from the inner dashed frontier to the outer solid one:

- **C: no reform.** Things stay as they are.
- **B: reform without compensation.** The public gains, but the elite loses: $u_r^B < u_r^C$.
- **A: reform with compensation.** The public transfers part of its gains to the elite. Both groups are better off than at C.

Four problems can prevent a deal at A or discourage investment:

1. **Incomplete information.** Neither side knows the other's willingness to concede, so bargaining delays reform [Alesina and Drazen, 1991].
2. **Compensation is infeasible.** Without transfers to reach A, reform means $C \rightarrow B$, making the elite an **economic loser**.
3. **Compensation is not credible.** Reform strengthens the public, which may then end transfers ($A \rightarrow B$). The elite blocks reform to avoid becoming a **political loser** [Acemoglu, 2003].
4. **Property protection is not credible.** The elite promises protection but may seize returns after investment. Anticipating this, people invest less; constraints on rulers can help [North and Weingast, 1989].

8 Ideas and Economic Growth

The question is what sustains growth at the frontier. Solow [1956] rules out capital accumulation alone because of diminishing returns. Knowledge can escape that constraint: an idea can be used repeatedly and by many producers at once.

8.1 AK and learning by doing

AK. Normalize fixed labor to one and let $\bar{a} > 0$ be constant. With saving rate s_K ,

$$\begin{aligned} Y &= \bar{a}K, \\ \dot{K} &= s_K Y - \delta K = (s_K \bar{a} - \delta)K, \\ g_Y = g_K &= \boxed{s_K \bar{a} - \delta}. \end{aligned} \tag{110}$$

There are no diminishing returns to aggregate capital. Hence a higher saving rate permanently raises growth; consumption $C = (1 - s_K)Y$ grows at the same rate once s_K is fixed.

Where might AK come from? Learning by doing raises productivity as firms invest and produce, as in airframe manufacturing and shipbuilding. In Romer [1986], firm i has $Y_i = F(K_i, A_i L_i)$, constant returns in private inputs, and a common spillover $A_i = bK$. With identical firms,

$$\begin{aligned} Y &= F(K, bKL) \\ &= K \underbrace{F(1, bL)}_{\bar{a}}. \end{aligned} \tag{111}$$

Fixed L makes $\bar{a}$ constant. Private firms take A_i as given; investment also benefits other firms, so its social return exceeds its private return.

The assumption is strong: learning and perfect spillovers must exactly offset diminishing returns. With $A_i = bK^\eta$ and Cobb–Douglas production,

$$\begin{aligned} Y &= b^{1-\alpha} L^{1-\alpha} K^{\alpha+\eta(1-\alpha)}, \\ g_K &= s_K \frac{Y}{K} - \delta \\ &= s_K b^{1-\alpha} L^{1-\alpha} K^{-(1-\alpha)(1-\eta)} - \delta. \end{aligned}$$

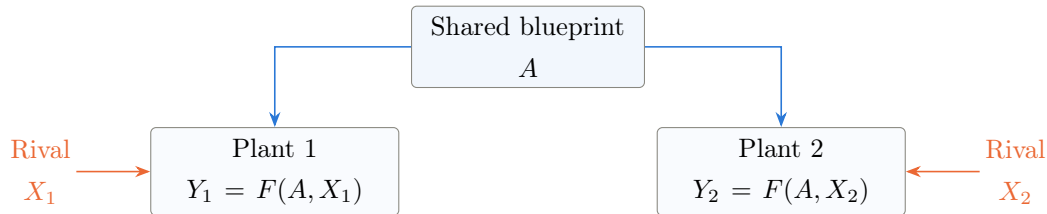
If $\eta = 1$ we recover AK; if $\eta < 1$, $g_K \rightarrow 0$ with fixed labor. These early externality models require strong accumulation assumptions, including in their human-capital versions. They do not isolate what makes ideas special.³⁹

³⁹Human capital is embodied in rival people. With fixed training, diminishing returns to learning and skill depreciation produce a finite skill level. Exponential growth requires learning to remain proportional to the existing skill stock and exceed depreciation; see Jones [2022, Section 2.3].

8.2 Nonrivalry and innovation incentives

Jones [2019] emphasizes the key distinction:

- **Objects are rival:** one machine or one worker's hour cannot serve two plants simultaneously.
- **Ideas are nonrival:** both plants can use the same blueprint, calculus, or production method.



Let X collect rival inputs and A index knowledge. Replicating a plant requires replicating X , while reusing A . For $m > 1$,

$$\begin{aligned} F(A, mX) &= mF(A, X), \\ F(mA, mX) &> mF(A, X). \end{aligned} \tag{112}$$

Thus production has constant returns to rival inputs and increasing returns to all inputs. In particular,

$$\begin{aligned} Y &= A^\sigma K^\alpha L^{1-\alpha}, \\ y = \frac{Y}{L} &= A^\sigma k^\alpha. \end{aligned}$$

Income per person depends on the total stock of knowledge, not knowledge per person. Nonrivalry is distinct from free access: patents can exclude users even though simultaneous use is feasible.

Who pays for invention? If competitive firms pay every rival input its marginal product, Euler's theorem gives

$$\sum_j F_{X_j} X_j = Y.$$

No revenue remains to cover discovery. In [Romer \[1990\]](#), imperfect competition generates profits that reward intentional R&D. Patent returns determine the incentive to conduct research.

8.3 Research, balanced growth, and convergence

Fix the research share $0 < s < 1$, as Solow [1956] fixes saving. There is no physical capital. The model is

$$\begin{aligned}
 Y &= A^\sigma L_Y && \text{goods production,} \\
 \dot{A} &= \theta L_A^\lambda A^\phi && \text{idea production,} \\
 L_A &= sL && \text{research labor,} \\
 L_Y &= (1-s)L && \text{goods labor,} \\
 L &= L_0 e^{nt} && \text{population.}
 \end{aligned} \tag{113}$$

- $\theta > 0$: research efficiency; $\sigma > 0$: elasticity of goods productivity to the stock of ideas.
- $\lambda > 0$: elasticity to researchers; $\lambda < 1$ allows duplication or congestion.
- ϕ : existing ideas facilitate discovery if $\phi > 0$; depletion of easy discoveries dominates if $\phi < 0$. Even $0 < \phi < 1$ makes **proportional** gains harder at fixed effort.

Divide by population and by the idea stock, respectively:

$$\begin{aligned}
 y &= (1-s)A^\sigma, \\
 g_y &= \sigma g_A, \\
 g_A &= \frac{\dot{A}}{A} = \theta (sL)^\lambda A^{\phi-1}.
 \end{aligned} \tag{114}$$

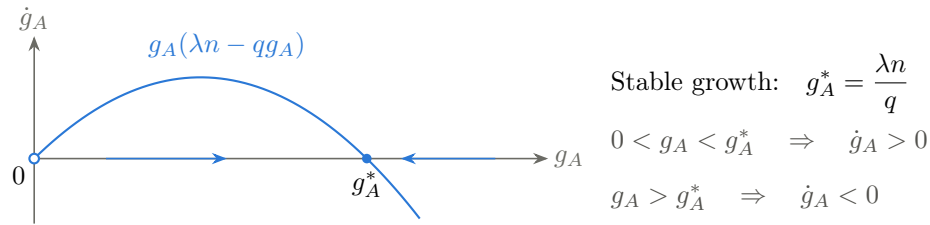
For the semi-endogenous case, define $q \equiv 1-\phi > 0$ and assume $n > 0$. Log-differentiating the last expression gives

$$\begin{aligned}
 \frac{\dot{g}_A}{g_A} &= \lambda n - q g_A, \\
 \dot{g}_A &= g_A(\lambda n - q g_A).
 \end{aligned} \tag{115}$$

On a balanced growth path (BGP), g_A is constant, so

$$\boxed{
 \begin{aligned}
 g_A^* &= \frac{\lambda n}{1-\phi}, \\
 g_y^* &= \frac{\sigma \lambda n}{1-\phi}.
 \end{aligned}
 } \tag{116}$$

More researchers offset the rising difficulty of proportional gains. The positive steady growth rate is stable: below g_A^* , research expands faster than difficulty and g_A rises; above it, g_A falls.



If $n = 0$ and $\phi < 1$, a fixed research workforce continues to discover ideas, but $g_A \rightarrow 0$. Positive constant growth requires exponentially growing research effort.⁴⁰

Three sources of sustained growth:

- **AK:** constant social returns make capital accumulation self-sustaining.
- **Romer [1990]:** research is linear in existing knowledge. Knowledge sustains growth even with fixed L_A ; research policy changes long-run growth (**fully endogenous**).
- **Jones [1995a]:** exponential population growth expands researchers when $\phi < 1$. Research is endogenous, but permanent growth depends on exogenous population growth (**semi-endogenous**).

8.4 Research policy and scale effects

Start on the BGP with $\phi < 1$. At t_0 , raise s_0 to $s_1 > s_0$. Since A is a state variable, it cannot jump, but discoveries can:

$$\begin{aligned} A(t_0^+) &= A(t_0^-), \\ \frac{g_A(t_0^+)}{g_A(t_0^-)} &= \left(\frac{s_1}{s_0}\right)^\lambda. \end{aligned} \quad (117)$$

After this initial jump, (115) returns g_A to the same g_A^* . Relative to unchanged policy (subscript b),

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{A_1(t)}{A_b(t)} &= \left(\frac{s_1}{s_0}\right)^{\lambda/q}, \\ \lim_{t \rightarrow \infty} \frac{y_1(t)}{y_b(t)} &= \frac{1-s_1}{1-s_0} \left(\frac{s_1}{s_0}\right)^{\sigma\lambda/q}. \end{aligned} \quad (118)$$

More research temporarily raises growth and permanently raises knowledge; it does not change the long-run growth rate when $\phi < 1$.

⁴⁰Differentiate A^q : its derivative is $q\theta(sL)^\lambda$, independent of A . Integrating makes A^q asymptotically exponential when $n > 0$, and linear when $n = 0$. The latter gives $g_A \rightarrow 0$.

Income initially falls by the factor $(1 - s_1)/(1 - s_0)$ because fewer workers produce goods. Its eventual level rises only if the productivity gain exceeds this resource cost; higher research is not automatically welfare improving.⁴¹

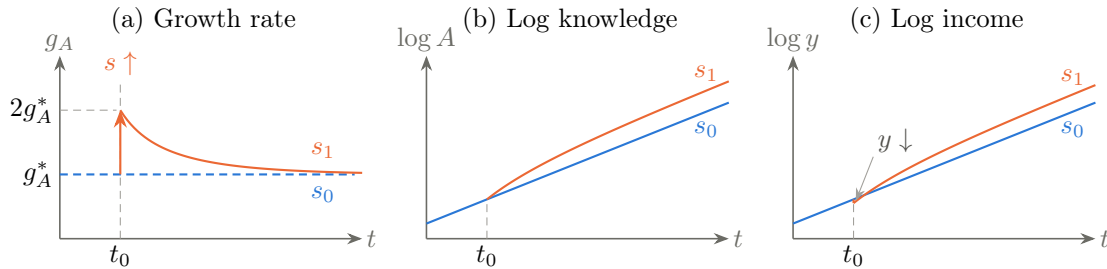


Figure 29: Higher research share: growth jumps; knowledge stays continuous; current goods production falls. Illustration: $s_0 = 0.1$, $s_1 = 0.2$, $q = \lambda = \sigma = 1$.

Strong versus weak scale effects.

$$\phi = 1 : \quad g_A = \theta(sL)^\lambda,$$

$$\phi < 1 : \quad g_y^* = \frac{\sigma \lambda n}{1 - \phi}.$$

- **Strong** ($\phi = 1$): More people or a larger research share raise growth. With $n > 0$, growth keeps accelerating rather than approaching a constant rate.
- **Weak** ($\phi < 1$): Population **growth** determines long-run growth; population **size** and research intensity affect levels.⁴²
- **Relevant scale:** the connected research economy, largely the world. Ideas cross borders, so a small country can use discoveries made elsewhere.
- **Empirical challenge:** research expanded while frontier growth stayed roughly stable [Jones, 1995b]. Strong scale effects would instead predict faster growth.

8.5 Are ideas getting harder to find?

Read growth and research effort together. Keep research efficiency θ constant and allow any $\lambda > 0$. Dividing idea production by A and log-differentiating gives

$$g_A = \theta L_A^\lambda A^{\phi-1},$$

$$\frac{d \log g_A}{dt} = \lambda g_{L_A} + (\phi - 1)g_A.$$

⁴¹Integrate the equation for A^q and compare the leading coefficients under the two research shares. Adjustment occurs at rate $\lambda n = qg_A^*$. Thus ϕ close to one implies slow adjustment **holding observed BGP growth fixed**, not holding n and λ fixed.

⁴²At a fixed BGP growth rate, the research equation makes A^q proportional to $(sL)^\lambda$. Insert that relation into goods production to obtain the income-level effect.

If positive idea growth stays constant while the research workforce expands,

$$\phi = 1 - \lambda \frac{g_{L_A}}{g_A} < 1.$$

Increasing effort offsets the rising difficulty of proportional gains. This does **not require** $\phi < 0$: useful knowledge spillovers can coexist with $\phi < 1$. The pattern does not identify λ and ϕ separately.

To measure ideas in productivity units, let $B = A^\sigma$ and $S = L_A$. Then

$$\begin{aligned} g_B &= \kappa S^\lambda B^{-\beta}, \\ \kappa &\equiv \sigma \theta, \\ \beta &\equiv \frac{1 - \phi}{\sigma}. \end{aligned} \tag{119}$$

For fixed s , $g_y = g_B$. Normalizing $\sigma = 1$ gives $B = A$ and $\beta = 1 - \phi$. [Bloom et al. \[2020\]](#) use the $\lambda = 1$ specification. Let $\mathcal{P}$ denote growth per researcher:

$$\begin{aligned} g_B &= S\mathcal{P}, \\ \mathcal{P} &= \frac{g_B}{S} = \kappa B^{-\beta}, \\ \frac{\dot{g}_B}{g_B} &= g_S - \beta g_B. \end{aligned} \tag{120}$$

Harder to find means more effort for the same **percentage** productivity gain; the stock of knowledge still rises.⁴³

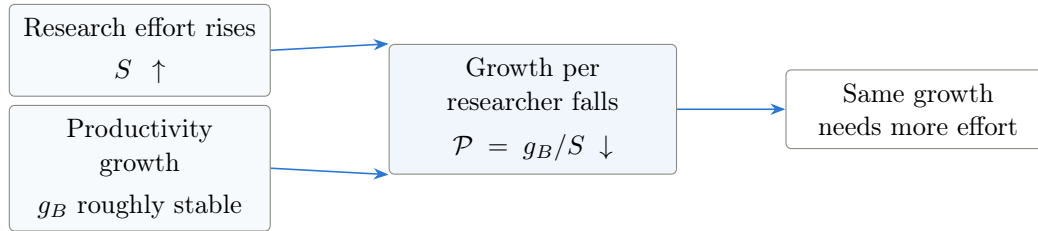
Measure effort in researchers, not dollars. Let R be real research expenditure and w the real skilled wage. In a lab-equipment model, multiply and divide by w :

$$\begin{aligned} \dot{B} &= \xi R, \\ g_B &= \frac{\xi R}{B} \\ &= \underbrace{\frac{\xi w}{B}}_{\text{research productivity}} \times \underbrace{\frac{R}{w}}_{\text{researcher equivalents}}. \end{aligned} \tag{121}$$

- **Input measure:** nominal R&D divided by a nominal skilled wage. Rising spending can reflect higher wages, without more researchers.
- **Benchmark:** in the fixed-population BGP of [Romer \[1990\]](#), both R/w and w/B can be constant.

⁴³With $\lambda \neq 1$, $g_{\mathcal{P}} = (\lambda - 1)g_S - \beta g_B$. Declining g_B/S alone then does not identify β : duplication can also lower growth per researcher.

The empirical message of [Bloom et al. \[2020\]](#):



- **Aggregate evidence:** sustaining productivity growth has required an expanding research workforce.
- **Moore's law:** continued chip improvements required more research effort. Steady technological progress need not mean steady research productivity.
- **Within technologies:** similar patterns appear in agriculture, health, and firm innovation. The aggregate finding is not simply a shift toward sectors with slower progress.
- **Interpretation:** the stock of knowledge keeps growing even as **percentage gains** become harder to obtain. Declining research productivity is widespread, but not universal.

Patents measure invention imperfectly.

- Counts treat breakthroughs and minor inventions alike; citations and market values try to distinguish their importance.
- [Kelly et al. \[2021\]](#) use text: a significant patent is **different from earlier patents** (novelty) and **similar to later patents** (influence).
- Innovation comes in waves and shifts across sectors. Patent activity is not itself a measure of proportional productivity growth.

8.6 Past growth, future growth, and missing inventors

[Jones \[2022\]](#) separates **long transitions** from the **permanent source of growth**. For the accounting, let N be population, L employment, $\ell = L/N$, and $y = Y/N$. Knowledge A and allocative efficiency M determine productivity $Z = AM$:

$$Y = K^\alpha [AMh(1-s)L]^{1-\alpha},$$

$$y = \left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)} AMh\ell(1-s).$$

Log differentiation separates the channels:

$$\begin{aligned} d \log y &= \frac{\alpha}{1-\alpha} d \log(K/Y) + d \log h + d \log \ell \\ &\quad + d \log(1-s) + d \log M + d \log A. \end{aligned} \tag{122}$$

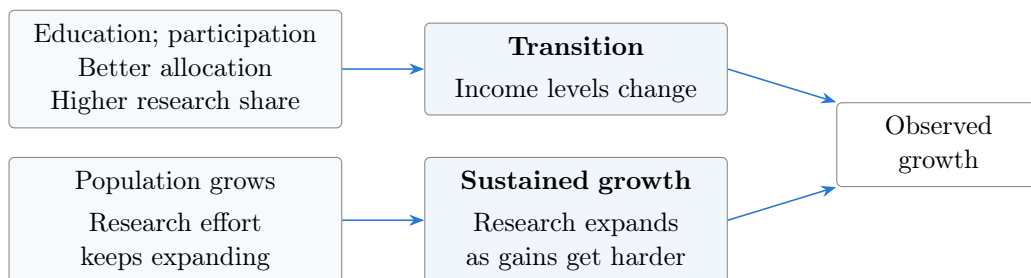
- **Capital, skills, and allocation:** higher K/Y , h , ℓ , or M raise income per person.
- **Research intensity:** a higher s reduces current goods production but builds knowledge faster.

Here A is labor-augmenting knowledge. With constant g_A , log differentiation of its production equation gives

$$\begin{aligned} 0 &= \lambda(g_s + g_L) - (1 - \phi)g_A, \\ g_A &= \gamma(g_s + g_L), \\ \gamma &\equiv \frac{\lambda}{1 - \phi}. \end{aligned}$$

Thus productivity growth in this accounting is⁴⁴

$$\begin{aligned} g_Z &= g_M + g_A \\ &= g_M + \gamma g_s + \gamma g_L. \end{aligned}$$



- **Past growth:** education, participation, allocation, and research intensity can account for substantial growth during long transitions. A level effect can be economically large.
- **Long run:** once these factors stabilize, their growth contributions disappear. With $\phi < 1$, sustained growth requires continually expanding research effort.
- **Demographic slowdown:** lower population growth eventually means slower expansion of researchers and slower idea growth, given the model's research technology.

What could improve the outlook?

- **Missing Einsteins:** poverty, caste/class barriers, and discrimination keep potential inventors out of research. Removing barriers increases the number and quality of researchers, hence discovery.
- **Automation and AI:** improving the technology of research can strengthen the feedback from ideas to new ideas. If that changes the $\phi < 1$ mechanism, the model's long-run growth conclusion can change. This is a mechanism to investigate, rather than a prediction of explosive growth.

⁴⁴Under the Cobb–Douglas convention, the Solow residual is $(1 - \alpha)g_Z$. Expressing output in terms of K/Y makes $g_Z = g_M + g_A$ the productivity contribution to income growth. The expression $g_A = \gamma(g_s + g_L)$ assumes constant g_A ; outside that case, replace the zero above by $\dot{g}_A/g_A$.

9 The Expanding Variety Model

Household saving and patent profits determine the allocation of labor to research, as in Romer [1990]; see Acemoglu [2009, Ch. 12.3–12.4 and 13.1–13.2].

- **Excludability:** patents, secrecy, or difficulty of learning can restrict access to nonrival knowledge. Reverse engineering and limits on patentability weaken exclusion.
- **Incentives:** **Efficient use of an existing idea calls for a zero access price, but creating it requires a reward.** A markup finances innovation while restricting use.
- **Building block:** Dixit and Stiglitz [1977] provide tractable monopolistic competition, also used in trade and New Keynesian models. Patent profits then make invention a private choice.

9.1 Dixit–Stiglitz: demand, prices, and variety

Dixit–Stiglitz assumptions

- A continuum of differentiated goods, indexed by $i \in [0, N]$; each good has a single supplier.
- Constant substitution elasticity $\varepsilon > 1$; each firm is negligible relative to the aggregate market.

Demand.

A household with expenditure Z maximizes the CES consumption index

$$\mathcal{C} = \left[\int_0^N c(i)^{(\varepsilon-1)/\varepsilon} di \right]^{\varepsilon/(\varepsilon-1)}, \quad \varepsilon > 1, \quad (123)$$

subject to the budget

$$\int_0^N p(i)c(i) di = Z.$$

Here ε is the elasticity of substitution. The index $\mathcal{C}$ differs from final-good consumption C below. The Lagrangian is

$$\mathcal{L} = \mathcal{C} - \lambda \left[\int_0^N p(i)c(i) di - Z \right].$$

The chain rule gives the FOC and individual demand:⁴⁵

$$\begin{aligned} \left(\frac{\mathcal{C}}{c(i)} \right)^{1/\varepsilon} &= \lambda p(i), \\ c(i) &= \mathcal{C} [\lambda p(i)]^{-\varepsilon}. \end{aligned} \tag{124}$$

Relative demand therefore satisfies

$$\frac{c(i)}{c(j)} = \left(\frac{p(i)}{p(j)} \right)^{-\varepsilon}.$$

Each negligible firm faces elasticity ε , taking aggregate spending and the price index as given.

The price index.

Collect the price terms in

$$B \equiv \int_0^N p(i)^{1-\varepsilon} di, \quad 0 < B < \infty.$$

Substitute individual demand into the CES index, then cancel the common consumption term:

$$\begin{aligned} \mathcal{C}^{(\varepsilon-1)/\varepsilon} &= \mathcal{C}^{(\varepsilon-1)/\varepsilon} \lambda^{1-\varepsilon} B, \\ 1 &= \lambda^{1-\varepsilon} B, \\ \lambda &= B^{1/(\varepsilon-1)}. \end{aligned}$$

The budget now becomes

$$\begin{aligned} Z &= \mathcal{C} \lambda^{-\varepsilon} B \\ &= \frac{\mathcal{C}}{\lambda}. \end{aligned}$$

Thus the unit expenditure index and demand are

$$P = \left[\int_0^N p(i)^{1-\varepsilon} di \right]^{1/(1-\varepsilon)}. \tag{125}$$

⁴⁵With a continuum, vary consumption on a small set of goods. The FOC holds almost everywhere. Dominated convergence permits differentiation under the integral when the nearby derivatives have an integrable bound; bounded perturbations away from zero consumption suffice for finite N .

$$Z = PC,$$

$$c(i) = C \left(\frac{p(i)}{P} \right)^{-\varepsilon}.$$

Interpretation: P is the minimum cost of one unit of the consumption index; real consumption equals spending divided by P .

The gain from variety.

Under symmetry, each good has the same price and quantity.⁴⁶ Thus

$$\begin{aligned} P &= pN^{-1/(\varepsilon-1)}, \\ C &= N^{\varepsilon/(\varepsilon-1)}c. \end{aligned} \tag{126}$$

Hold the total physical quantity $Q = Nc$ fixed. Then

$$C = Q N^{1/(\varepsilon-1)}.$$

- More varieties deliver more services for the same expenditure or physical quantity.
- Higher substitutability reduces this gain: similar goods add less than distinct ones.

Monopoly pricing.

Manufacturing technology

- Producing one unit of any good uses $\psi > 0$ units of the numeraire.

Marginal cost is therefore constant.⁴⁷ A monopolist chooses its price to maximize

$$\Pi(p) = (p - \psi)C \left(\frac{p}{P} \right)^{-\varepsilon}.$$

Differentiate, cancel the positive demand term, and rearrange:

$$\begin{aligned} 1 - \varepsilon \frac{p - \psi}{p} &= 0, \\ (\varepsilon - 1)p &= \varepsilon\psi. \end{aligned}$$

⁴⁶Identical marginal costs and the same demand elasticity produce the same optimal markup, derived below. Symmetric demand then implies equal quantities. This is an equilibrium consequence of common primitives, not a separate restriction on firms' choices.

⁴⁷Constant marginal cost comes from linear manufacturing: producing twice as much uses twice as many resources. This is a modeling assumption, not an implication of CES demand.

Hence the constant markup is

$$p = \frac{\varepsilon}{\varepsilon - 1} \psi. \quad (127)$$

More elastic demand lowers the markup.⁴⁸ The same rule prices CES intermediate inputs.

9.2 Romer: the economy and its flows

The model of [Romer \[1990\]](#) makes the research allocation endogenous, instead of fixing the research share exogenously:

$$L_R = sL.$$

More research sacrifices goods today to create productive varieties tomorrow.

Environment

- A fixed workforce L can produce final goods or conduct research.
- A positive stock of designs N_0 is initially available. A design can be used repeatedly; manufactured intermediates are used up each period.
- The final good is the numeraire. There is no accumulated physical capital, and households own all firms in equal proportions.

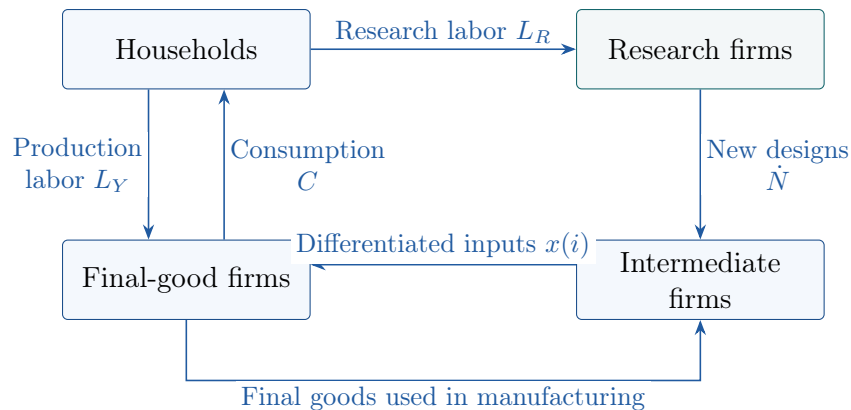


Figure 30: Real flows in the Romer economy. Households own all firms; wages and patent income finance consumption and saving. Research creates designs, while manufacturing uses final goods to produce the corresponding inputs.

⁴⁸With positive marginal cost, an elasticity at or below one has no finite interior profit-maximizing price. Above one, profit is zero at marginal cost and tends to zero as price tends to infinity; the FOC selects the interior maximum.

9.3 Households

Household preferences

- Labor supply is inelastic. Households save through financial claims on firms and consume the final good.
- Preferences have discount rate $\rho > 0$ and curvature $\theta > 0$.

With aggregate consumption C and financial wealth $\mathcal{W}$, the problem is

$$\max_{\{C(t)\}} \int_0^\infty e^{-\rho t} u(C(t)) dt,$$

where

$$u(C) = \begin{cases} \frac{C^{1-\theta}}{1-\theta}, & \theta \neq 1, \\ \log C, & \theta = 1. \end{cases}$$

The financial budget is

$$\dot{\mathcal{W}} = r\mathcal{W} + wL - C.$$

Optimization gives⁴⁹

$$\boxed{\frac{\dot{C}}{C} = \frac{r - \rho}{\theta}}. \quad (128)$$

- A return above impatience rewards postponing consumption.
- Greater curvature makes households less willing to shift consumption across dates. Sustaining a given growth rate then requires a higher return.

9.4 Final-good producers

Final production

- Final-good firms are competitive and take wages and input prices as given.
- Production combines labor with differentiated intermediate goods.

⁴⁹The current-value Hamiltonian and its first-order conditions are

$$\begin{aligned} \mathcal{H}_h &= u(C) + \mu_h(r\mathcal{W} + wL - C), \\ \mu_h &= C^{-\theta}, \\ \dot{\mu}_h &= (\rho - r)\mu_h. \end{aligned}$$

Log-differentiating the consumption condition gives the displayed Euler equation. Impose a no-Ponzi restriction and household transversality.

Competitive firms use labor and a measure N of intermediate varieties:

$$Y = \frac{L_Y^\beta}{1 - \beta} \int_0^N x(i)^{1-\beta} di, \quad 0 < \beta < 1. \quad (129)$$

For given N , returns are constant in the rival inputs. The CES link is

$$\mathcal{X} = \left[\int_0^N x(i)^{1-\beta} di \right]^{1/(1-\beta)},$$

$$Y = \frac{L_Y^\beta \mathcal{X}^{1-\beta}}{1 - \beta}, \quad \varepsilon = \frac{1}{\beta}.$$

Additive separability lets new varieties coexist with old ones, unlike quality ladders. A design may represent a product, machine, or process.

With the final good as numeraire, the firm maximizes

$$Y - \int_0^N p(i)x(i) di - wL_Y.$$

The input FOC gives inverse demand and then quantity demanded:

$$\begin{aligned} p(i) &= L_Y^\beta x(i)^{-\beta}, \\ x(i) &= p(i)^{-1/\beta} L_Y. \end{aligned} \quad (130)$$

The labor FOC and total input payments imply

$$\begin{aligned} wL_Y &= \beta Y, \\ \int_0^N p(i)x(i) di &= (1 - \beta)Y. \end{aligned}$$

These payments exhaust output: competitive final producers earn zero profit.

9.5 Intermediate-good producers

Patents and manufacturing

- A successful inventor has a perpetual monopoly on its variety.
- All varieties have the same manufacturing technology: one unit costs ψ units of final output.

Given the demand from final producers, a patent owner maximizes operating profit. Substituting that demand gives

$$\Pi(p) = (p - \psi)p^{-1/\beta}L_Y.$$

The derivative is

$$\Pi'(p) = \frac{L_Y}{\beta}p^{-1/\beta-1}[\psi - (1 - \beta)p].$$

Setting it to zero recovers the CES markup with elasticity $1/\beta$:

$$p = \frac{\psi}{1 - \beta}. \quad (131)$$

Normalize the manufacturing technology as⁵⁰

$$\psi = 1 - \beta.$$

Identical costs and elasticities imply the same price and quantity for every variety:

$$\begin{aligned} p(i, t) &= 1, \\ x(i, t) &= L_Y(t). \end{aligned}$$

Substitution into production and factor payments yields

$$\begin{aligned} Y &= \frac{NL_Y}{1 - \beta}, \\ w &= \frac{\beta N}{1 - \beta}, \\ \Pi &= (p - \psi)x = \beta L_Y. \end{aligned} \quad (132)$$

- **Aggregate productivity:** more designs raise output and wages.
- **Per-patent profits:** at fixed production labor, an incumbent's sales and profits stay constant even as the economy grows.

⁵⁰For arbitrary ψ , equilibrium output, wages, and profits share the same positive factor:

$$\left(\frac{1 - \beta}{\psi}\right)^{(1 - \beta)/\beta}.$$

It cancels from the growth equations. This cost normalization is separate from the final-good numeraire. A unit price still exceeds marginal cost, so monopoly profits remain positive.

9.6 Research firms and patent values

The value of a patent.

A patent buys the future operating profits of one variety:

$$V(t) = \int_t^\infty \exp\left[-\int_t^s r(v) dv\right] \Pi(s) ds. \quad (133)$$

Its return is dividends plus capital gains:⁵¹

$$rV = \Pi + \dot{V}.$$

For constant profits and a constant positive interest rate,

$$V = \frac{\Pi}{r} = \frac{\beta L_Y}{r}.$$

Research technology.

Innovation and entry

- Research uses labor. Existing knowledge raises each researcher's productivity, with a unit elasticity in the stock of designs.
- Entry into invention is free. New varieties coexist with old ones; innovation does not destroy an incumbent patent's market.

Researchers create designs according to

$$\dot{N} = \eta N L_R, \quad \eta > 0. \quad (134)$$

Each researcher produces ηN designs per unit of time. Free entry with positive research equates patent revenue to the wage.⁵²

$$\eta N V = w.$$

⁵¹Leibniz's rule gives the asset equation. Dominated convergence justifies differentiation when discounted profits and the relevant derivatives have an integrable bound. Recovering the present-value formula from the differential equation also requires the discounted terminal patent value to vanish (no bubble).

⁵²At zero research, the expected patent revenue per researcher cannot exceed the wage. Equality is required only for an interior research allocation.

Use the wage from (132):

$$\begin{aligned} V &= \frac{w}{\eta N} \\ &= \frac{\beta}{(1 - \beta)\eta}. \end{aligned} \tag{135}$$

Thus patent value is constant under interior entry. The asset equation then determines the private return.⁵³

$$\boxed{r = \frac{\Pi}{V} = (1 - \beta)\eta L_Y.}$$

- **Demand for invention:** more production labor raises the market for each variety and the private return to research.
- **Entry:** inventing earns zero net profit; an existing patent still earns positive operating profits.
- **Spillover:** a discovery raises future researchers' productivity without compensating its inventor.

9.7 Market clearing

Labor and the final good satisfy

$$\begin{aligned} L &= L_Y + L_R, \\ Y &= C + \psi \int_0^N x(i) \, di. \end{aligned}$$

Using the symmetric input quantity, net consumption is

$$\begin{aligned} C &= Y - \psi \int_0^N x(i) \, di \\ &= \frac{NL_Y}{1 - \beta} - (1 - \beta)NL_Y \\ &= \kappa^M NL_Y, \\ \kappa^M &\equiv \frac{\beta(2 - \beta)}{1 - \beta}. \end{aligned} \tag{136}$$

- Subtract manufacturing costs, rather than sales revenue: monopoly profits are income, not resources lost.

⁵³More final production raises sales and profit per patent. The cost of inventing one design is constant: both the research wage and patents per researcher rise proportionally with the stock of ideas. Higher production labor therefore raises the profit-to-value ratio.

- Research uses labor. Its resource cost is forgone production, with no additional final-good expense.⁵⁴

9.8 Balanced growth

A balanced growth path has constant production and research labor. Consumption and gross output are proportional to the stock of ideas:

$$g \equiv \frac{\dot{N}}{N} = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C}.$$

The system is

$$\begin{aligned} r &= \rho + \theta g && \text{(household Euler equation),} \\ r &= (1 - \beta)\eta L_Y && \text{(private return to research),} \\ L_Y + L_R &= L && \text{(labor feasibility),} \\ g &= \eta L_R && \text{(idea production).} \end{aligned} \tag{137}$$

Eliminate research labor and production labor:

$$L_Y = L - \frac{g}{\eta}.$$

Substitute into the two return equations, then collect growth terms:⁵⁵

$$\begin{aligned} \rho + \theta g &= (1 - \beta)(\eta L - g), \\ (\theta + 1 - \beta)g &= (1 - \beta)\eta L - \rho. \end{aligned}$$

Allowing for zero research gives

$$g^M = \max \left\{ 0, \frac{(1 - \beta)\eta L - \rho}{\theta + 1 - \beta} \right\}. \tag{138}$$

⁵⁴The household budget agrees: wealth is the stock of patents times their value, and payments for new patents finance research wages. Combining the asset equation with these payments leaves consumption equal to production wages plus operating profits.

⁵⁵Faster growth raises the return households require by θ per unit of growth. It also moves labor out of production, reducing the private return by $1 - \beta$. Both adjustments close the gap between the initial research return and impatience.

The labor allocation follows directly:

$$L_R^M = \frac{g^M}{\eta},$$

$$L_Y^M = L - \frac{g^M}{\eta}.$$

Research is positive when

$$\rho < (1 - \beta)\eta L.$$

Otherwise, all labor produces goods, growth is zero, and the interest rate equals impatience.⁵⁶

Comparative statics.

- **Research productivity and patience:** higher η or lower ρ increases positive growth. Larger θ reduces it.
- **Scale versus level: Population raises the growth rate; the initial stock of ideas raises only the level of the path.**
- **Aggregate versus individual:** wages and total patent wealth grow with the stock of ideas. Profit and value per patent stay constant.

9.9 Optimal allocation

1. Use existing varieties efficiently.

At given ideas and production labor, choose intermediate quantities to maximize net output:

$$\max_{\{x(i)\}} \int_0^N \left[\frac{L_Y^\beta}{1 - \beta} x(i)^{1-\beta} - \psi x(i) \right] di.$$

Set marginal product equal to resource cost:

$$L_Y^\beta (x^S)^{-\beta} = \psi,$$

$$x^S = (1 - \beta)^{-1/\beta} L_Y. \tag{139}$$

⁵⁶ Assume parameters satisfying the transversality and finite-utility restriction. For the market-planner comparison it suffices that $\rho > (1 - \theta)\eta L$; this is automatic for $\theta \geq 1$ and ensures positive production labor at the interior planner allocation.

Use symmetry to obtain gross output, then net consumption:

$$\begin{aligned}
 Y^S &= (1 - \beta)^{-1/\beta} N L_Y, \\
 C^S &= Y^S - \psi N x^S \\
 &= \kappa^S N L_Y, \\
 \kappa^S &\equiv \beta(1 - \beta)^{-1/\beta}.
 \end{aligned} \tag{140}$$

Static efficiency: holding production labor fixed, the planner uses more of every intermediate and delivers more consumption:

$$x^S > x^M, \quad \kappa^S > \kappa^M.$$

The full equilibrium also changes the allocation of labor.

2. Express the dynamic constraint in terms of consumption.

Efficient production requires

$$L_Y = \frac{C}{\kappa^S N}.$$

Substitute this into the research technology:

$$\begin{aligned}
 \dot{N} &= \eta N (L - L_Y) \\
 &= \eta L N - \frac{\eta}{\kappa^S} C.
 \end{aligned} \tag{141}$$

Consumption must leave a feasible, nonnegative research workforce:

$$0 < C \leq \kappa^S N L.$$

Economic meaning: the social return to knowledge is ηL . Holding consumption fixed, an extra design both raises research productivity and frees production labor for research.

3. Derive optimal growth.

The current-value Hamiltonian is

$$\mathcal{H} = u(C) + \mu \left[\eta L N - \frac{\eta}{\kappa^S} C \right].$$

At an interior optimum, the consumption FOC gives

$$C^{-\theta} = \frac{\eta}{\kappa^S} \mu,$$

$$\mu = \frac{\kappa^S}{\eta} C^{-\theta}.$$

The costate equation is

$$\dot{\mu} = \rho\mu - \frac{\partial \mathcal{H}}{\partial N}$$

$$= (\rho - \eta L)\mu.$$

Log-differentiate the consumption FOC and use the costate equation:

$$-\theta \frac{\dot{C}}{C} = \frac{\dot{\mu}}{\mu},$$

$$-\theta \frac{\dot{C}}{C} = \rho - \eta L.$$

Thus the feasible optimal balanced growth rate is⁵⁷

$$g^S = \max \left\{ 0, \frac{\eta L - \rho}{\theta} \right\}. \quad (142)$$

The associated labor allocation is

$$L_R^S = \frac{g^S}{\eta},$$

$$L_Y^S = L - \frac{g^S}{\eta}.$$

9.10 Economic conclusions

With positive entry, the private return falls short of the planner's return:

$$\underbrace{(1 - \beta)\eta L_Y}_{\text{private return}} < \underbrace{\eta L}_{\text{social return}}.$$

⁵⁷Integrate the linear state equation along the optimal consumption path. Transversality removes its explosive homogeneous component and fixes initial consumption so that ideas grow at the same rate. The parameter restriction above ensures positive production labor. If impatience exceeds the social return, all labor produces goods and the stock of ideas remains constant.

Whenever socially optimal research is positive,

$$g^M < g^S, \quad L_R^M < L_R^S.$$

The market allocates too little labor to research and grows too slowly.⁵⁸

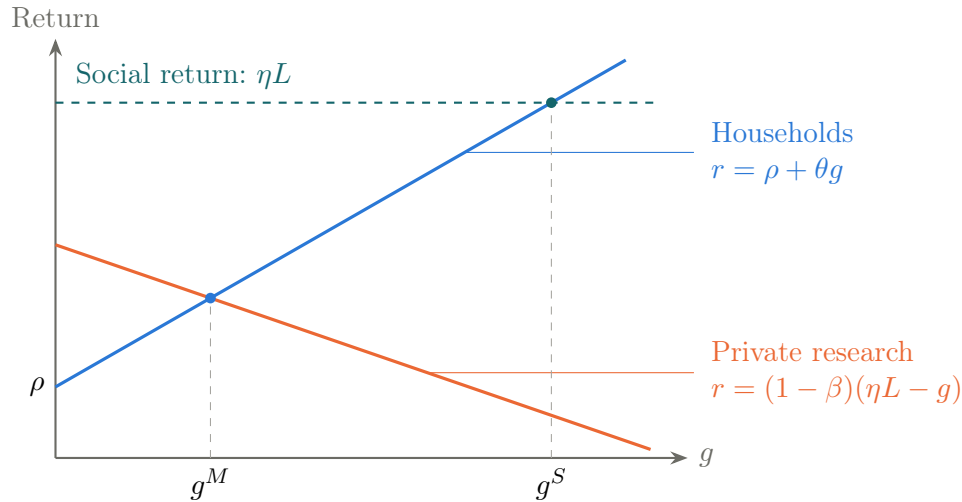


Figure 31: Balanced growth equates the household return to the relevant return on research. Faster market growth both requires a higher saving return and withdraws labor from production. The planner internalizes the full return to knowledge. Interior allocations shown schematically.

Three distortions explain the allocation:

- **Incomplete appropriation:** patent profits omit benefits to users and workers, weakening the incentive to invent.
- **Research spillovers:** discoveries help later inventors without payment to their creators.
- **Monopoly pricing:** markups restrict current input use. Even perpetual patents leave all three failures in place.

Policy and welfare.

- **Reward discovery:** research subsidies, public research (NIH, NSF), prizes, and recognition can increase innovation.
- **Improve access:** subsidies to patented goods can reduce the markup distortion while preserving inventors' revenue. Financing, project selection, and demand elasticities matter for implementation.

⁵⁸There can be positive socially optimal research even when the market conducts none. If impatience exceeds the social return, both growth rates are zero; the static monopoly distortion remains.

- **Evaluate the whole consumption path: Higher growth is not itself a welfare criterion.** Cheaper drugs help current users but may reduce future invention. Patience and the value of new discoveries determine the tradeoff; more research can lower initial consumption.

Actual patents are also temporary and imperfectly enforceable.

Changing how research uses resources.

- **Semi-endogenous:** a knowledge elasticity below one gives

$$\dot{N} = \eta N^\phi L_R, \quad \phi < 1.$$

With fixed research labor, the growth rate tends to zero. Population growth can sustain growth.

- **Lab equipment:** research uses final goods Z , while fixed labor L produces output. With the same pricing normalization and an imposed research share s_R of gross output,

$$\begin{aligned} \dot{N} &= \eta Z, \\ Y &= bN, \quad b \equiv \frac{L}{1-\beta}, \\ Z &= s_R Y = s_R b N, \\ g_N &= \frac{\dot{N}}{N} = \eta s_R b. \end{aligned}$$

A positive feasible s_R sustains exponential growth and permanently raises its rate when increased.⁵⁹ Under the semi-endogenous technology, a higher fixed research labor share raises the level of the long-run knowledge path without changing balanced growth. Final-good research can finance either new varieties or better quality.

From new varieties to better versions.

- **Horizontal innovation** adds product lines i ; existing producers keep their sales and patents.
- **Vertical innovation** improves version j within a fixed line. The entrant takes the incumbent's sales and rents: this **business stealing** separates private rewards from social gains.

⁵⁹The resource constraint is $C + \psi \int_0^N x(i) di + Z = Y$, so $C/Y = 1 - s_R - (1 - \beta)^2$. Positive consumption requires $0 \leq s_R < \beta(2 - \beta)$. The fixed share is an imposed allocation; optimizing households and free entry need not choose that share.

10 Creative Destruction: The Quality Ladder Model

Growth can come from improving an existing product, with the new version replacing its predecessor. This is **creative destruction** in the innovation tradition of [Schumpeter \[1934\]](#). The quality-ladder model of [Aghion and Howitt \[1992\]](#) formalizes this mechanism; the specification here uses final goods to finance research [[Acemoglu, 2009](#), Ch. 14.1].

10.1 Product lines, versions, and physical quantities

Products and quality

- Product lines form a fixed continuum $i \in [0, 1]$; innovation raises quality within a line rather than adding new lines.
- A successful innovation multiplies quality by $\lambda > 1$. Different versions within the same line are perfect substitutes after adjusting for their effectiveness.
- Labor supply L is fixed. All labor produces final goods; research instead uses final-good expenditure.

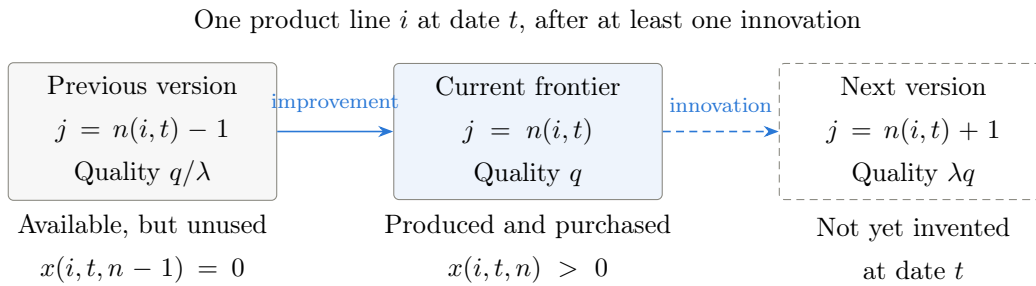
Read the three arguments in $x(i, t, j)$ separately:

- i identifies a **product line**: a particular type of intermediate input.
- t is the **date** at which production and purchases occur.
- j identifies a **quality version** within line i . It is a discrete rung, not another product line or a date.
- $x(i, t, j)$ is the **physical quantity** of version j purchased at date t . Quality and physical quantity are different objects.

Let $n(i, t)$ count successful innovations in line i by date t . For a positive initial quality $q_0(i)$,

$$\begin{aligned} q_j(i) &= \lambda^j q_0(i), & j &= 0, 1, \dots, n(i, t), \\ q(i, t) &= q_{n(i, t)}(i) = \lambda^{n(i, t)} q_0(i). \end{aligned} \tag{143}$$

The subscript in $q_j(i)$ labels the version; the exponent in λ^j counts the multiplicative improvements. The current frontier is $q(i, t)$.



Convert physical units into productive services.

The production coefficient q_j is normalized so that one physical unit of version j supplies $q_j^{1/(1-\beta)}$ effective units. Thus the quality-adjusted services of line i are

$$h(i, t) = \sum_{j=0}^{n(i, t)} q_j(i)^{1/(1-\beta)} x(i, t, j), \quad 0 < \beta < 1. \quad (144)$$

Final production aggregates these services across different lines:

$$Y(t) = \frac{L^\beta}{1-\beta} \int_0^1 h(i, t)^{1-\beta} di. \quad (145)$$

- The **sum over** j combines discrete versions of the same input, linearly in effective units: they are perfect substitutes.
- The **integral over** i combines distinct product lines. Diminishing returns within each line make these inputs imperfect substitutes, with CES elasticity $1/\beta$.
- An old version remains **technologically available** after improvement. In equilibrium it can be **economically obsolete**: no buyer chooses it at a price covering its production cost.

10.2 Households and final-goods demand

Households and markets

- Households own firms, supply labor inelastically, and can diversify across product lines. Idiosyncratic innovation risk washes out in the aggregate; there is no aggregate uncertainty.
- Final goods are competitively produced and serve as numeraire. Preferences have discount rate $\rho > 0$ and curvature $\theta > 0$.
- Initial aggregate quality is positive and finite; consumption paths satisfy feasibility, no-Ponzi, and transversality conditions.

Preferences and the aggregate financial budget are

$$\begin{aligned} U &= \int_0^\infty e^{-\rho t} u(C(t)) dt, \\ u'(C) &= C^{-\theta}, \\ \dot{W} &= rW + wL - C. \end{aligned}$$

Log utility is the case $\theta = 1$. Household optimization yields

$$\boxed{g_C = \frac{r - \rho}{\theta}}. \quad (146)$$

A higher return makes delayed consumption more attractive.⁶⁰

Demand when only the frontier version is used.

Write its physical quantity as

$$x(i, t) \equiv x(i, t, n(i, t)).$$

Substituting its services into production gives

$$\begin{aligned} h(i, t) &= q(i, t)^{1/(1-\beta)} x(i, t), \\ Y(t) &= \frac{L^\beta}{1-\beta} \int_0^1 q(i, t) x(i, t)^{1-\beta} di. \end{aligned} \quad (147)$$

Equivalently, define the quality-weighted CES composite

$$\begin{aligned} \mathcal{X}(t) &= \left[\int_0^1 q(i, t) x(i, t)^{1-\beta} di \right]^{1/(1-\beta)}, \\ Y(t) &= \frac{L^\beta \mathcal{X}(t)^{1-\beta}}{1-\beta}. \end{aligned}$$

⁶⁰The current-value consumption FOC equates marginal utility to the wealth costate. The costate grows at rate $\rho - r$; log differentiation gives Euler's equation. Diversified households receive the expected return on patent wealth, including capital losses when leaders are replaced.

A competitive final producer maximizes output minus intermediate and labor payments. Its first-order conditions are

$$\begin{aligned} L^\beta q(i, t) x(i, t)^{-\beta} &= p(i, t), \\ x(i, t) &= \left(\frac{q(i, t)}{p(i, t)} \right)^{1/\beta} L, \\ w(t)L &= \beta Y(t). \end{aligned} \tag{148}$$

Quantity responds to quality relative to price. A larger q raises demand at a fixed physical-unit price. The next step checks when older versions actually permit the leader to charge its monopoly price.

10.3 Monopoly pricing or limit pricing?

Version-specific production

- A patent gives exclusive rights to its own version. It does not prevent another firm from inventing the next version.
- Manufacturing one physical unit of quality q costs ψq units of the final good, where $\psi > 0$.
- Suppliers of available versions compete in prices. A version is used only if it offers the lowest price per effective unit in its line.

1. Compute the unconstrained monopoly price.

Suppress line and date indices. If rivals impose no binding constraint,

$$\Pi(p) = (p - \psi q) \left(\frac{q}{p} \right)^{1/\beta} L.$$

Holding the version's quality fixed, differentiate with respect to price:

$$\begin{aligned} \Pi'(p) &= \frac{Lq^{1/\beta}}{\beta} p^{-1/\beta-1} [\psi q - (1 - \beta)p], \\ \Pi'(p) = 0 &\implies p^M(q) = \frac{\psi q}{1 - \beta}. \end{aligned}$$

The price is a constant markup over physical-unit marginal cost.

2. Compare versions in common service units.

Buyers compare $p_j/q_j^{1/(1-\beta)}$, not physical prices alone. For a given line service level, the marginal products satisfy

$$\frac{\partial Y}{\partial x(i, t, j)} = L^\beta h(i, t)^{-\beta} q_j(i)^{1/(1-\beta)},$$

$$\frac{\text{MP}_{\text{old}}}{\text{MP}_{\text{new}}} = \left(\frac{q/\lambda}{q} \right)^{1/(1-\beta)} = \lambda^{-1/(1-\beta)}.$$

The predecessor has quality q/λ and marginal cost $\psi q/\lambda$. For the leader to be no more expensive per effective unit,

$$\frac{p}{q^{1/(1-\beta)}} \leq \frac{\psi q/\lambda}{(q/\lambda)^{1/(1-\beta)}},$$

$$p \leq \psi q \lambda^{\beta/(1-\beta)}.$$

Hence its **limit price** is

$$p^{\text{lim}}(q) = \psi q \lambda^{\beta/(1-\beta)}. \quad (149)$$

The second-best version is the relevant competitor: even older versions have higher cost per effective unit.⁶¹

3. Determine which constraint binds.

$$p(q) = \min \left\{ \frac{\psi q}{1-\beta}, \psi q \lambda^{\beta/(1-\beta)} \right\}. \quad (150)$$

The unconstrained monopoly price is feasible precisely when

$$\frac{1}{1-\beta} \leq \lambda^{\beta/(1-\beta)},$$

$$\lambda \geq \left(\frac{1}{1-\beta} \right)^{(1-\beta)/\beta}.$$

- **Large improvement:** the leader's quality advantage is enough to eliminate competition even at the monopoly price.
- **Small improvement:** the predecessor constrains the price. A lower markup still lets the better version capture the market.

⁶¹Effective marginal cost is $\psi q_j^{-\beta/(1-\beta)}$, which falls as quality rises. At the exact limit price, buyers are indifferent between the leader and a predecessor pricing at cost; assign ties to the leader, or interpret limit pricing as an arbitrarily small undercut.

10.4 Who innovates, and what grows?

An entrant receives the whole value of a new leadership position. An incumbent improving its own product also gives up its old profit stream:

$$\begin{aligned}\text{entrant's gain} &= V(\lambda a), \\ \text{incumbent's gain} &= V(\lambda a) - V(a),\end{aligned}$$

where a denotes quality before the improvement. This **replacement effect** weakens the incumbent's incentive. With identical research costs and free entry, an incumbent cannot match an entrant's willingness to spend. Incumbent innovation can reappear with cost advantages or strategic commitment that discourages entry [Barro and Sala-i Martin, 2004, Ch. 7.1]; see also Acemoglu [2009, Ch. 14.3–14.4].

Benchmark used to solve for growth

- Improvements strictly exceed the threshold for unconstrained monopoly pricing, so the predecessor cannot force a lower price.
- New firms perform the innovation and displace incumbent leaders.
- Choose units so that $\psi = 1 - \beta$.

Price, physical quantity, and operating profit become

$$\begin{aligned}p(i, t) &= q(i, t), \\ x(i, t) &= L, \\ \Pi(i, t) &= [p(i, t) - \psi q(i, t)]x(i, t) \\ &= \beta q(i, t)L.\end{aligned}\tag{151}$$

Growth comes from better inputs, even though their physical quantity per product line stays constant. Define aggregate quality by

$$Q(t) = \int_0^1 q(i, t) di.\tag{152}$$

Substituting $x(i, t) = L$ into production gives

$$\begin{aligned}Y(t) &= \frac{Q(t)L}{1 - \beta}, \\ w(t) &= \frac{\beta Q(t)}{1 - \beta}.\end{aligned}\tag{153}$$

- The measure of lines remains one. Q is their aggregate quality, not a count of goods or the physical sum of intermediates.
- Quality raises aggregate productivity and wages. Within one patent's lifetime, its own quality and profit remain unchanged.

10.5 Innovation risk and the value of a patent

Research expenditure determines a hazard.

When current quality in line i is $a = q(i, t)$, total research expenditure $Z(i, t)$ generates arrival intensity

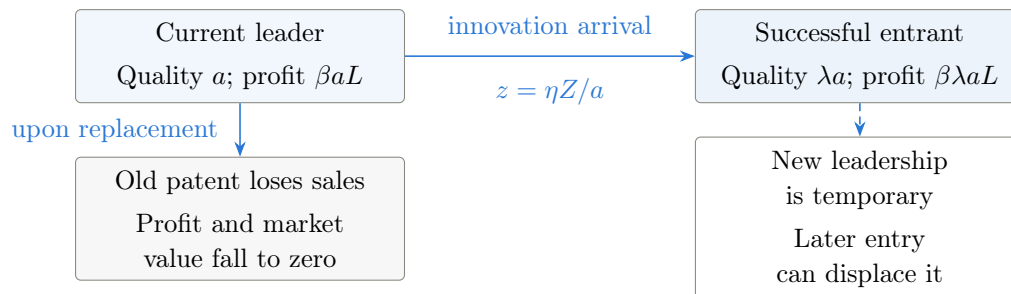
$$z(i, t) = \frac{\eta Z(i, t)}{a}, \quad \eta > 0. \quad (154)$$

Over a short interval dt ,

$$\Pr(\text{one innovation}) = z(i, t) dt + o(dt),$$

$$\Pr(\text{no innovation}) = 1 - z(i, t) dt + o(dt).$$

- z is an **arrival rate**, not the number of certain discoveries per unit of time. A successful arrival moves quality to λa .
- Higher quality requires more expenditure for the same arrival rate. But each successful step is a larger absolute improvement.
- Innovation times and $n(i, t)$ are random in each line; aggregate quality is deterministic under the diversification assumption.



A patent survives only until the next successful entrant.

On a balanced growth path let the displacement hazard be the constant z^* . The probability that a current leader survives another u units of time is

$$S(u) = e^{-z^* u}.$$

A patent of current quality q earns βqL while it survives and zero afterward. With constant r ,

$$\begin{aligned} V(q) &= \int_0^\infty e^{-ru} S(u) \beta qL \, du \\ &= \int_0^\infty e^{-(r+z^*)u} \beta qL \, du \\ &= \boxed{\frac{\beta qL}{r+z^*}}. \end{aligned} \tag{155}$$

Destruction acts like extra discounting. The owner requires a return for both time and the risk of losing leadership.⁶²

10.6 Free entry and balanced growth

1. Keep quality before and after success distinct.

One extra unit spent when current quality is a buys extra arrival intensity η/a . Success produces a patent of quality λa . With positive research, free entry therefore requires

$$\begin{aligned} 1 &= \frac{\eta}{a} V(\lambda a) \\ &= \frac{\eta}{a} \frac{\beta \lambda a L}{r+z^*}, \\ r+z^* &= \lambda \eta \beta L. \end{aligned} \tag{156}$$

Equivalently, if q labels the **successful new version**, its predecessor was q/λ :

$$\frac{\eta}{q/\lambda} V(q) = 1.$$

The factor λ comes from valuing the **new** quality while research costs depend on the **old** quality.

2. Translate arrivals into aggregate growth.

A successful step adds $(\lambda - 1)q(i, t)$ to line quality. Thus

$$\dot{Q}(t) = (\lambda - 1) \int_0^1 z(i, t) q(i, t) \, di.$$

⁶²The small-interval asset equation is $rV = \Pi + \partial_t V - zV$: destruction reduces the patent's value to zero. On the BGP, a surviving version has fixed q , so $\partial_t V = 0$. Do not subtract aggregate growth from the denominator: an individual surviving patent's profit does not grow. Nonnegative discounted profits permit exchanging expectation and the value integral by Tonelli's theorem.

When all lines face the same constant hazard z^* ,

$$\begin{aligned} Q(t + dt) &= Q(t) + z^* dt (\lambda - 1) Q(t) + o(dt), \\ \dot{Q}(t) &= (\lambda - 1) z^* Q(t), \\ g_Q &= g_Y = (\lambda - 1) z^*. \end{aligned} \tag{157}$$

Growth equals arrival frequency times the proportional step. A larger λ makes each success more consequential.⁶³

3. Account for the goods used in research.

Let total research expenditure be $R(t) = \int_0^1 Z(i, t) di$. The common hazard implies

$$R(t) = \frac{z^*}{\eta} Q(t).$$

Final output pays for consumption, intermediate manufacturing, and R&D:

$$\begin{aligned} C + \psi \int_0^1 q(i, t) x(i, t) di + R &= Y, \\ C &= \left[\frac{\beta(2 - \beta)}{1 - \beta} L - \frac{z^*}{\eta} \right] Q. \end{aligned} \tag{158}$$

With constant z^* and a positive consumption coefficient, C , Y , and Q grow at the same rate g .

4. Solve the equilibrium system.

The three conditions are

$$\begin{aligned} r + z^* &= \lambda \eta \beta L && \text{free entry,} \\ r &= \rho + \theta g && \text{household Euler equation,} \\ g &= (\lambda - 1) z^* && \text{quality accumulation.} \end{aligned}$$

Substitute the second and third conditions into the first:

$$\begin{aligned} \rho + \theta(\lambda - 1) z^* + z^* &= \lambda \eta \beta L, \\ [1 + \theta(\lambda - 1)] z^* &= \lambda \eta \beta L - \rho. \end{aligned}$$

⁶³The quality index averages levels. Mean log quality instead grows at rate $z^* \log \lambda$; that is not the growth rate of Q . The aggregation step uses diversified line-level shocks and finite initial aggregate quality.

Allowing for the zero-research corner,

$$\boxed{\begin{aligned} z^* &= \max \left\{ 0, \frac{\lambda\eta\beta L - \rho}{1 + \theta(\lambda - 1)} \right\}, \\ g^M &= (\lambda - 1)z^*. \end{aligned}} \quad (159)$$

The equivalent interior growth formula is

$$g^M = \frac{\lambda\eta\beta L - \rho}{\theta + (\lambda - 1)^{-1}}.$$

- **Positive innovation:** requires $\lambda\eta\beta L > \rho$. Otherwise $z^* = g = 0$ and $r = \rho$; the return to research does not exceed its cost.
- **Comparative statics:** higher research efficiency η , a larger workforce L , or greater patience raise positive growth. Larger improvements λ also raise it; a larger θ lowers it.
- **Scale effect:** population size raises growth, while initial aggregate quality shifts the level of the path.

Require finite lifetime utility and transversality:

$$\rho > (1 - \theta)g^M.$$

This is automatic for $\theta \geq 1$. It also ensures positive equilibrium consumption.⁶⁴

10.7 Welfare: the private prize and the social gain

Creative destruction creates opposing incentives:

- **Incomplete appropriation:** the patent owner does not capture all benefits to users and workers, weakening research incentives.
- **Knowledge passed forward:** each improved version becomes the starting point for later improvements. Social benefits can survive the current patent's destruction.
- **Business stealing:** a new leader receives a whole profit stream, including rents taken from the old leader. Transferring those rents is not itself an increase in aggregate welfare.

The entrant values the whole new patent; society values the improvement after accounting for what it replaces. These opposing forces make the sign of the

⁶⁴Euler's equation makes the restriction equivalent to $r > g^M$. Using free entry, the consumption coefficient can be written $\beta L / (1 - \beta) + (r - g^M) / (\lambda\eta) > 0$. At the zero-research corner, consumption is the positive net output from manufacturing.

innovation distortion a question about the model's primitives. In particular, the research technology here does **not** make a dollar of R&D more productive when aggregate quality is higher: the larger step and the lower hazard per dollar exactly offset.

A planner comparison for this benchmark.

The planner first eliminates monopoly distortions. Equate a version's marginal product to its manufacturing cost:

$$\begin{aligned} qL^\beta(x^S)^{-\beta} &= \psi q, \\ x^S &= (1 - \beta)^{-1/\beta} L. \end{aligned}$$

Net output available for consumption and research is

$$\begin{aligned} C + R &= \left[\frac{L^\beta(x^S)^{1-\beta}}{1 - \beta} - \psi x^S \right] Q \\ &= \frac{\beta}{1 - \beta} \psi x^S Q \\ &= \kappa^S L Q, \\ \kappa^S &\equiv \beta(1 - \beta)^{-1/\beta}. \end{aligned}$$

The hazard technology implies, for any allocation of research across lines,

$$\begin{aligned} \dot{Q} &= (\lambda - 1) \int_0^1 q(i, t) \frac{\eta Z(i, t)}{q(i, t)} di \\ &= (\lambda - 1) \eta R. \end{aligned}$$

Write $\omega \equiv (\lambda - 1)\eta$. Eliminate research expenditure:

$$\dot{Q} = \omega \kappa^S L Q - \omega C. \tag{160}$$

This is a linear accumulation problem. The current-value Hamiltonian and its interior conditions are

$$\begin{aligned} \mathcal{H} &= u(C) + \mu[\omega \kappa^S L Q - \omega C], \\ C^{-\theta} &= \omega \mu, \\ \dot{\mu} &= (\rho - \omega \kappa^S L) \mu. \end{aligned}$$

Log-differentiating the consumption FOC gives

$$-\theta g_C = \rho - \omega \kappa^S L.$$

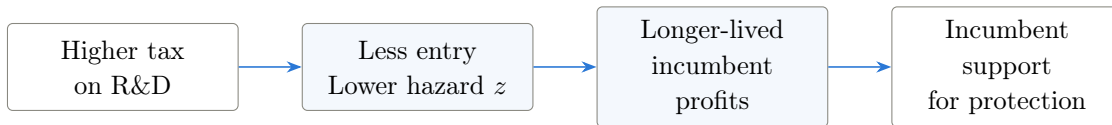
The planner's growth rate is therefore⁶⁵

$$g^S = \max \left\{ 0, \frac{(\lambda - 1)\eta\kappa^S L - \rho}{\theta} \right\}. \quad (161)$$

- **Under the maintained large-improvement assumption**, the full planner correcting input use chooses more innovation whenever the market innovates. The monopoly and appropriation losses dominate business stealing in this specification.⁶⁶
- **In broader quality-ladder models**, business stealing can instead dominate and innovation can be excessive. The existence of creative destruction alone therefore does not establish whether growth is too low or too high [[Acemoglu, 2009](#), Ch. 14.1.4].

10.8 Why incumbents may oppose innovation

An R&D tax reduces the private payoff per research dollar. Entry falls, so the arrival rate of replacements falls.



- **Incumbents gain protection:** their current profit stream lasts longer when challengers innovate less frequently.
- **Political incentive:** established leaders may lobby for policies that slow entry and aggregate growth.
- **Distribution matters:** innovation creates gains for users and entrants while destroying incumbent rents. Support for growth need not be unanimous, even when the economy would benefit overall.

⁶⁵For an interior optimum, impose $\rho > (1 - \theta)(\lambda - 1)\eta\kappa^S L$, automatically true if $\theta \geq 1$. Integrating the linear state equation and applying transversality fixes the initial consumption level and gives common consumption and quality growth. If the social return is no greater than ρ , the planner does no research.

⁶⁶At the large-step pricing threshold, the social return $(\lambda - 1)\eta\kappa^S L$ already exceeds $\lambda\eta\beta L$; the inequality strengthens with the step size. Substituting these returns into the two growth formulas establishes the comparison.

References

- ABRAMOVITZ, M. (1956): “Resource and Output Trends in the United States since 1870,” *American Economic Review*, 46, 5–23.
- ACEMOGLU, D. (2003): “Why Not a Political Coase Theorem? Social Conflict, Commitment, and Politics,” *Journal of Comparative Economics*, 31, 620–652.
- (2009): *Introduction to Modern Economic Growth*, Princeton: Princeton University Press.
- ACEMOGLU, D. AND D. AUTOR (2011): “Skills, Tasks and Technologies: Implications for Employment and Earnings,” in *Handbook of Labor Economics*, ed. by O. Ashenfelter and D. Card, Elsevier, vol. 4B, 1043–1171.
- ACEMOGLU, D., S. JOHNSON, AND J. ROBINSON (2005): “The Rise of Europe: Atlantic Trade, Institutional Change, and Economic Growth,” *American Economic Review*, 95, 546–579.
- ACEMOGLU, D., S. JOHNSON, AND J. A. ROBINSON (2001): “The Colonial Origins of Comparative Development: An Empirical Investigation,” *American Economic Review*, 91, 1369–1401.
- (2012): “The Colonial Origins of Comparative Development: An Empirical Investigation: Reply,” *American Economic Review*, 102, 3077–3110.
- ACEMOGLU, D. AND P. RESTREPO (2018): “The Race between Man and Machine: Implications of Technology for Growth, Factor Shares, and Employment,” *American Economic Review*, 108, 1488–1542.
- AGHION, P. AND P. HOWITT (1992): “A Model of Growth Through Creative Destruction,” *Econometrica*, 60, 323–351.
- ALBOUY, D. Y. (2012): “The Colonial Origins of Comparative Development: An Empirical Investigation: Comment,” *American Economic Review*, 102, 3059–3076.
- ALESINA, A. AND A. DRAZEN (1991): “Why Are Stabilizations Delayed?” *American Economic Review*, 81, 1170–1188.
- ALSAN, M. (2015): “The Effect of the TseTse Fly on African Development,” *American Economic Review*, 105, 382–410.
- ARCHIBONG, B., B. S. COULIBALY, AND N. OKONJO-IWEALA (2021): “Washington Consensus Reforms and Lessons for Economic Performance in Sub-Saharan Africa since the 1980s,” *Journal of Economic Perspectives*, 35, 133–156.
- BARRO, R. J. (1991): “Economic Growth in a Cross Section of Countries,” *Quarterly Journal of Economics*, 106, 407–443.

- BARRO, R. J. AND J.-W. LEE (2013): “A New Data Set of Educational Attainment in the World, 1950–2010,” *Journal of Development Economics*, 104, 184–198.
- BARRO, R. J. AND X. SALA-I MARTIN (1991): “Convergence across States and Regions,” *Brookings Papers on Economic Activity*, 1991, 107–182.
- (1992): “Convergence,” *Journal of Political Economy*, 100, 223–251.
- (2004): *Economic Growth*, Cambridge, MA: MIT Press, 2nd ed.
- BAUMOL, W. J. (1967): “Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis,” *American Economic Review*, 57, 415–426.
- (1986): “Productivity Growth, Convergence, and Welfare: What the Long-Run Data Show,” *American Economic Review*, 76, 1072–1085.
- BERGEAUD, A., G. CETTE, AND R. LECAT (2016): “Productivity Trends in Advanced Countries between 1890 and 2012,” *Review of Income and Wealth*, 62, 420–444.
- BIVENS, J. AND L. MISHEL (2015): “Understanding the Historic Divergence between Productivity and a Typical Worker’s Pay,” Briefing Paper 406, Economic Policy Institute.
- BLOOM, N., C. I. JONES, J. VAN REENEN, AND M. WEBB (2020): “Are Ideas Getting Harder to Find?” *American Economic Review*, 110, 1104–1144.
- BRYAN, G., S. CHOWDHURY, AND A. M. MOBARAK (2014): “Underinvestment in a Profitable Technology: The Case of Seasonal Migration in Bangladesh,” *Econometrica*, 82, 1671–1748.
- CASELLI, F. (2005): “Accounting for Cross-Country Income Differences,” in *Handbook of Economic Growth*, ed. by P. Aghion and S. N. Durlauf, Elsevier, vol. 1A, 679–741.
- CASELLI, F. AND J. FEYRER (2007): “The Marginal Product of Capital,” *Quarterly Journal of Economics*, 122, 535–568.
- CHANCEL, L. AND T. PIKETTY (2021): “Global Income Inequality, 1820–2020: The Persistence and Mutation of Extreme Inequality,” *Journal of the European Economic Association*, 19, 3025–3062.
- CHETTY, R., N. HENDREN, AND L. F. KATZ (2016): “The Effects of Exposure to Better Neighborhoods on Children: New Evidence from the Moving to Opportunity Experiment,” *American Economic Review*, 106, 855–902.
- CLARK, G. (2010): “The Macroeconomic Aggregates for England 1209–1869,” *Research in Economic History*, 27, 51–140.

- DAVID, P. A. (1977): “Invention and Accumulation in America’s Economic Growth: A Nineteenth-Century Parable,” *Carnegie-Rochester Conference Series on Public Policy*, 6, 179–228.
- DE LONG, J. B. (1988): “Productivity Growth, Convergence, and Welfare: Comment,” *American Economic Review*, 78, 1138–1154.
- DIXIT, A. K. AND J. E. STIGLITZ (1977): “Monopolistic Competition and Optimum Product Diversity,” *American Economic Review*, 67, 297–308.
- EASTERLY, W. (2002): *The Elusive Quest for Growth*, Cambridge, MA: MIT Press.
- ENGERMAN, S. L. AND K. L. SOKOLOFF (1997): “Factor Endowments, Institutions, and Differential Paths of Growth Among New World Economies: A View from Economic Historians of the United States,” in *How Latin America Fell Behind*, ed. by S. Haber, Stanford, CA: Stanford University Press, 260–304.
- FEENSTRA, R. C., R. INKLAAR, AND M. P. TIMMER (2015): “The Next Generation of the Penn World Table,” *American Economic Review*, 105, 3150–3182.
- FERNALD, J. (2012): “A Quarterly, Utilization-Adjusted Series on Total Factor Productivity,” Working Paper 2012-19, Federal Reserve Bank of San Francisco.
- FERNALD, J. AND B. NEIMAN (2011): “Growth Accounting with Misallocation: Or, Doing Less with More in Singapore,” *American Economic Journal: Macroeconomics*, 3, 29–74.
- GLAESER, E. L., R. LA PORTA, F. LOPEZ-DE SILANES, AND A. SHLEIFER (2004): “Do Institutions Cause Growth?” *Journal of Economic Growth*, 9, 271–303.
- GOLDIN, C. AND L. F. KATZ (2008): *The Race between Education and Technology*, Cambridge, MA: Harvard University Press.
- GOLLIN, D. (2002): “Getting Income Shares Right,” *Journal of Political Economy*, 110, 458–474.
- HALL, R. E. AND C. I. JONES (1999): “Why Do Some Countries Produce So Much More Output per Worker than Others?” *Quarterly Journal of Economics*, 114, 83–116.
- HARCOURT, G. C. (1969): “Some Cambridge Controversies in the Theory of Capital,” *Journal of Economic Literature*, 7, 369–405.
- HASSANEIN, A. AND J. STEINSSON (2026): “A Simple Introduction to Task-Based Models of Technical Change,” University of California, Berkeley.
- HENDRICKS, L. AND T. SCHOELLMAN (2018): “Human Capital and Development Accounting: New Evidence from Wage Gains at Migration,” *Quarterly Journal of Economics*, 133, 665–700.

- HOUTHAKKER, H. S. (1955): “The Pareto Distribution and the Cobb-Douglas Production Function in Activity Analysis,” *Review of Economic Studies*, 23, 27–31.
- HSIEH, C.-T. (2002): “What Explains the Industrial Revolution in East Asia? Evidence from the Factor Markets,” *American Economic Review*, 92, 502–526.
- HSIEH, C.-T. AND P. J. KLENOW (2010): “Development Accounting,” *American Economic Journal: Macroeconomics*, 2, 207–223.
- INADA, K.-I. (1964): “On a Two-Sector Model of Economic Growth: Comments and a Generalization,” *Review of Economic Studies*, 30, 119–127.
- JONES, B. F. AND X. LIU (2024): “A Framework for Economic Growth with Capital-Embodied Technical Change,” *American Economic Review*, 114, 1448–1487.
- JONES, B. F. AND B. A. OLKEN (2005): “Do Leaders Matter? National Leadership and Growth since World War II,” *Quarterly Journal of Economics*, 120, 835–864.
- JONES, C. I. (1995a): “R&D-Based Models of Economic Growth,” *Journal of Political Economy*, 103, 759–784.
- (1995b): “Time Series Tests of Endogenous Growth Models,” *Quarterly Journal of Economics*, 110, 495–525.
- (2016): “The Facts of Economic Growth,” in *Handbook of Macroeconomics*, ed. by J. B. Taylor and H. Uhlig, Elsevier, vol. 2A, 3–69.
- (2019): “Paul Romer: Ideas, Nonrivalry, and Endogenous Growth,” *Scandinavian Journal of Economics*, 121, 859–883.
- (2021): *Macroeconomics*, New York, NY: W. W. Norton, 5 ed.
- (2022): “The Past and Future of Economic Growth: A Semi-Endogenous Perspective,” *Annual Review of Economics*, 14, 125–152.
- JONES, C. I. AND C. TONETTI (2026): “Past Automation and Future A.I.: How Weak Links Tame the Growth Explosion,” Working paper.
- JONES, E. L. (1981): *The European Miracle: Environments, Economies and Geopolitics in the History of Europe and Asia*, Cambridge, UK: Cambridge University Press.
- JORGENSEN, D. W. AND Z. GRILICHES (1967): “The Explanation of Productivity Change,” *Review of Economic Studies*, 34, 249–283.
- KALDOR, N. (1961): “Capital Accumulation and Economic Growth,” in *The Theory of Capital*, ed. by F. A. Lutz and D. C. Hague, London: Macmillan, 177–222.
- KARABARBOUNIS, L. AND B. NEIMAN (2014): “The Global Decline of the Labor Share,” *Quarterly Journal of Economics*, 129, 61–103.

- KATZ, L. F. AND K. M. MURPHY (1992): “Changes in Relative Wages, 1963–1987: Supply and Demand Factors,” *Quarterly Journal of Economics*, 107, 35–78.
- KELLY, B., D. PAPANIKOLAOU, A. SERU, AND M. TADDY (2021): “Measuring Technological Innovation over the Long Run,” *American Economic Review: Insights*, 3, 303–320.
- KLENOW, P. J. AND A. RODRÍGUEZ-CLARE (1997): “The Neoclassical Revival in Growth Economics: Has It Gone Too Far?” *NBER Macroeconomics Annual*, 12, 73–103.
- KOH, D., R. SANTAELIÀ-LLOPIS, AND Y. ZHENG (2020): “Labor Share Decline and Intellectual Property Products Capital,” *Econometrica*, 88, 2609–2628.
- KREMER, M., J. WILLIS, AND Y. YOU (2021): “Converging to Convergence,” *NBER Macroeconomics Annual*, 36, 337–412.
- LUCAS, ROBERT E., J. (1990): “Why Doesn’t Capital Flow from Rich to Poor Countries?” *American Economic Review*, 80, 92–96.
- MANKIW, N. G., D. ROMER, AND D. N. WEIL (1992): “A Contribution to the Empirics of Economic Growth,” *Quarterly Journal of Economics*, 107, 407–437.
- NAKAMURA, E., J. SIGURDSSON, AND J. STEINSSON (2022): “The Gift of Moving: Intergenerational Consequences of a Mobility Shock,” *Review of Economic Studies*, 89.
- NORTH, D. C. (1990): *Institutions, Institutional Change and Economic Performance*, Cambridge, UK: Cambridge University Press.
- NORTH, D. C. AND R. P. THOMAS (1973): *The Rise of the Western World: A New Economic History*, Cambridge, UK: Cambridge University Press.
- NORTH, D. C. AND B. R. WEINGAST (1989): “Constitutions and Commitment: The Evolution of Institutions Governing Public Choice in Seventeenth Century England,” *Journal of Economic History*, 49, 803–832.
- OBERFIELD, E. AND D. RAVAL (2021): “Micro Data and Macro Technology,” *Econometrica*, 89, 703–732.
- OBSTFELD, M. (1992): “Dynamic Optimization in Continuous-Time Economic Models (A Guide for the Perplexed),” Manuscript, University of California, Berkeley.
- PHILIPPON, T. (2023): “Additive Growth,” Working paper, National Bureau of Economic Research.

- PINKOVSKIY, M., X. SALA-I MARTIN, K. CHATTERJI-LEN, AND W. H. NOBER (2024): “Inequality within Countries Is Falling: Underreporting-Robust Estimates of World Poverty, Inequality and the Global Distribution of Income,” *NBER Working Paper* 32058.
- PRITCHETT, L. (1997): “Divergence, Big Time,” *Journal of Economic Perspectives*, 11, 3–17.
- PSACHAROPOULOS, G. (1994): “Returns to Investment in Education: A Global Update,” *World Development*, 22, 1325–1343.
- RESTUCCIA, D., D. T. YANG, AND X. ZHU (2008): “Agriculture and Aggregate Productivity: A Quantitative Cross-Country Analysis,” *Journal of Monetary Economics*, 55, 234–250.
- ROMER, D. (2019): *Advanced Macroeconomics*, New York, NY: McGraw-Hill, 5 ed.
- ROMER, P. M. (1986): “Increasing Returns and Long-Run Growth,” *Journal of Political Economy*, 94, 1002–1037.
- (1990): “Endogenous Technological Change,” *Journal of Political Economy*, 98, S71–S102.
- ROY, A. D. (1951): “Some Thoughts on the Distribution of Earnings,” *Oxford Economic Papers*, 3, 135–146.
- SACHS, J. D. (2003): “Institutions Don’t Rule: Direct Effects of Geography on Per Capita Income,” Working Paper 9490, National Bureau of Economic Research.
- SALA-I MARTIN, X. (1997): “I Just Ran Two Million Regressions,” *American Economic Review*, 87, 178–183.
- (2006): “The World Distribution of Income: Falling Poverty and... Convergence, Period,” *Quarterly Journal of Economics*, 121, 351–397.
- SAMUELSON, P. A. (1966): “A Summing Up,” *Quarterly Journal of Economics*, 80, 568–583.
- SCHUMPETER, J. A. (1934): *The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle*, Cambridge, MA: Harvard University Press.
- SMITH, M., D. YAGAN, O. ZIDAR, AND E. ZWICK (2022): “The Rise of Pass-Throughs and the Decline of the Labor Share,” *American Economic Review: Insights*, 4, 323–340.
- SOLOW, R. M. (1956): “A Contribution to the Theory of Economic Growth,” *Quarterly Journal of Economics*, 70, 65–94.

- (1957): “Technical Change and the Aggregate Production Function,” *Review of Economics and Statistics*, 39, 312–320.
- TINBERGEN, J. (1974): “Substitution of Graduate by Other Labour,” *Kyklos*, 27, 217–226.
- TOMBE, T. (2015): “The Missing Food Problem: Trade, Agriculture, and International Productivity Differences,” *American Economic Journal: Macroeconomics*, 7, 226–258.
- UZAWA, H. (1961): “Neutral Inventions and the Stability of Growth Equilibrium,” *Review of Economic Studies*, 28, 117–124.
- WILLIAMSON, J. G. (1995): “The Evolution of Global Labor Markets since 1830: Background Evidence and Hypotheses,” *Explorations in Economic History*, 32, 141–196.
- YOUNG, A. (1995): “The Tyranny of Numbers: Confronting the Statistical Realities of the East Asian Growth Experience,” *Quarterly Journal of Economics*, 110, 641–680.
- ZEIRA, J. (1998): “Workers, Machines, and Economic Growth,” *Quarterly Journal of Economics*, 113, 1091–1117.