About Brouwer fixed point theorem and its applications in general equilibrium

Marcelo Gallardo*
Carlos Cosentino*
Eduardo Llamoca*

*Mathematics, Pontificia Universidad Católica del Perú, Lima, Perú

E-mails: marcelo.gallardo@pucp.edu.pe
a20203544@pucp.edu.pe
a20200397@pucp.edu.pe

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We develop a path towards the proof of Brouwer's Fixed Point Theorem. We also present an application in economic theory: the existence of the Walrasian Equilibrium. We aim to provide the simplest. or at least one of the simplest, proof for Brouwer's Fixed Point Theorem. The only requirements are real analysis and general topology. Besides one Lemma which is not proved in its most general case, we prove all the results building up to the main theorem. It is important to emphasize that this work does not seek to be original or innovative, we do not introduce any new results in the literature. Our goal is to develop a clear and understandable approach to Brouwer's Fixed Point Theorem and its applications in general equilibrium.

1 Introduction

In economic theory, Brouwer Fixed Point Theorem is a very powerful tool, establishing very important results in, for instance, general equilibrium theory. This field of microeconomic theory has been one of the most captivating and significant topics in economic theory. Mathematical formalism elegantly intertwines with economics, which accounts for the numerous works, including both books and articles, that delve into this subject, for instance Echenique and Wierman (2012), Echenique (2023) or Ok (2007). There are numerous extensions and generalizations, as is the case with infinite consumption goods, the indivisibility of goods, etc Aliprantis et al. (1990). When we reach the applications part of this document, we mainly follow the classical scenario presented in Mas-Colell et al. (1995) or Ellickson (1993). We do not treat the general cases or extensions involving, for instance, elements of functional analysis.

The structure of this document is as follows. First, in Section 2 we present Brouwer Fixed Point Theorem (BFPT) and work our way up to the proof. For this, we require a strong Lemma known as «Borsuk Lemma». We provide a rigorous proof of this lemma, based on the sktech provided in Laczkovich and Sos (2017). Although it is not the most general statement of Borsuk lemma, it is one that avoids passing through algebraic topology. Thereafter, we state and prove some additional results. Only equipped with this, we are ready to tackle the main result. Finally, in Section 3, we move to the application of BFPT in general equilibrium.

2 An elementary proof of Brouwer Fixed Point Theorem

The statement of Brouwer Fixed Point Theorem is very short and elegant. Nevertheless, its proof is not simple at all. Let us first announce this famous result.

Theorem 1. *Let X be a non empty convex and compact subset of $\mathbb{R}^n$ and $f : X \to X$ a continuous function. Then, there exists $x^* \in X$ such that $f(x^*) = x^*$.*

Theorem 1 establishes therefore the existence of a fixed point for a continuous function going from a convex and compact subset in $\mathbb{R}^n$ to itself.

To prove this result we start defining what a retraction is, in the context of general topology. After this, we state Borsuk lemma and provide a rigorous proof for a weaker case. Once this lemma is established, Theorem 5 arises as a direct application. We then present and prove several results that allow us to reach the BFPT.

Definition 2. Let Y be a topological space and $S \subseteq Y$ be a subset of Y equipped with the subspace topology. A continuous function $r : Y \to S$ is called a retraction if $r(x) = x$ for all $x \in S$. In other words, a retraction is a continuous function $r : Y \to S$ that fixes S. When such a function exists we say that S is a retract of Y.

Lemma 3. Borsuk. The n dimensional closed unit ball $\overline{\mathbb{B}}^n$ does not retract to the $n-1$ dimensional unit sphere $\mathbb{S}^{n-1}$.

The proof of Lemma 3 involves, most of the time, passing through algebraic topology. See for instance Boothby (1971). In this work, we prove that there is no continuously differentiable retraction from $\overline{\mathbb{B}}^n$ to $\mathbb{S}^{n-1}$: a weaker statement since the general framework is for continuous maps.

Proposition 4. No differentiable retraction. There is no mapping f such that

1. f is continuously differentiable on an open set containing $\overline{\mathbb{B}}^n$
2. $f(\overline{\mathbb{B}}^n) = \mathbb{S}^{n-1}$
3. $f(x) = x$ for all $x \in \mathbb{S}^{n-1}$.

Proof. We proceed by contradiction. Take f satisfying all these three conditions. For each $t \in [0, 1]$, define the mapping f_t by

$$f_t(x) = (1-t)f(x) + tx.$$

Clearly, $f_0 = f$, f_1 is the identity and each f_t satisfies conditions (1) and (3). Furthermore, $f_t(\overline{\mathbb{B}}^n) \subset \overline{\mathbb{B}}^n$ due to the convexity of $\overline{\mathbb{B}}^n$. Consider the function $h : [0, 1] \to \mathbb{R}$ given by

$$h(t) = \int_{\mathbb{B}^n} \det f_t'(x) dx.$$

We show that:

(i) $h(0) = 0$

(ii) $h(t)$ is a polynomial

(iii) $h(t) = m(\mathbb{B}^n)$ hold in an interval $(1-\delta, 1]$ for some $\delta > 0$. Here $m(\mathbb{B}^n)$ denotes the Lebesgue measure of the unit ball.

The contradiction is hard to miss: a polynomial which is constant in an open interval is a constant itself, and cannot take both the values of 0 and $m(\mathbb{B}^n) > 0$.

For (i), write $f = (f_1, \ldots, f_n)$ and note that condition (2) can be rephrased as

$$||f(x)||^2 = \sum_{i=1}^{n} f_i^2(x) = 1 \quad \forall x \in \mathbb{B}^n.$$

Differentiate to get

$$\sum_{i=1}^{n} 2 f_i(x) \nabla f_i(x) = 0 \quad \forall x \in \mathbb{B}^n.$$

That is, for each $x \in \mathbb{B}^n$ the row vectors $\{\nabla f_i(x)\}_{1 \le i \le n}$ of $f_t'(x)$ are linearly dependent and thus $\det f_t'(x) = 0$. Integrate on the unit ball to get $h(0) = 0$.

For (ii), observe that $f_t'(x) = (1-t) f'(x) + tI$, where $I \in \mathcal{M}_{n \times n}$ is the identity matrix. Then, the entries of $f_t'(x)$ are

$$(f_t'(x))_{ij} = \begin{cases} (1-t) \frac{\partial f_i}{\partial x_j}(x) + t, \ i = j \\ (1-t) \frac{\partial f_i}{\partial x_j}(x), \ i \neq j, \end{cases}$$

which are continuous in x due to condition (1). It is not hard to see that $\det f_t'(x)$ has the form $\sum_{i=1}^{m} s_i(t) \cdot g_i(x)$, for some s_i polynomials and g_i continuous functions. In particular, for each i, g_i is continuous in $\overline{\mathbb{B}}^n$ and hence bounded in the same region. Set $c_i = \int_{\mathbb{B}^n} g_i(x) dx \in \mathbb{R}$ for each i. Integrate $\det f_t'(x)$ on the unit ball finally to obtain that $h(t)$ is the polynomial $\sum_{i=1}^{m} c_i s_i(t)$.

For (iii), recall that, as each f_t satisfies condition (1), a change of variables allows

$$\int_{\mathbb{B}^n} |\det f_t'(x)| dx = m(f_t(\mathbb{B}^n)),$$

as long as f_t is injective in $\mathbb{B}^n$. Hence, it suffices to prove, for t sufficiently close to 1, that f_t is injective in $\mathbb{B}^n$, $\det f_t'(x) > 0$ for all $x \in \mathbb{B}^n$ and $f_t(\mathbb{B}^n) = \mathbb{B}^n$. We go in order.

As f is continuously differentiable on the compact $\overline{\mathbb{B}^n}$, it is Lipschitz of constant $M > 0$. That is, $||f(x) - f(y)|| \leq M||x - y||$ for all $x, y \in \overline{\mathbb{B}^n}$. Let $x, y \in \mathbb{B}^n$ with $x \neq y$. For t sufficiently close to 1 we get $t - (1 - t)M > 0$ and injectivity is then assured:

$$\begin{aligned} ||f_t(x) - f_t(y)|| &\geq t||x - y|| - (1 - t)||f(x) - f(y)|| \\ &\geq t||x - y|| - (1 - t)M||x - y|| \\ &= (t - (1 - t)M)||x - y|| > 0. \end{aligned}$$

Now consider the space $\mathcal{M}_{n\times n}$ with the norm $||\cdot||_\infty$. Given that the fuction $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$ is a polynomial on its coordinates, it is continuous. The identity, I, has determinant equal to 1, then there exists $\xi > 0$ such that $||A - I||_\infty < \xi$ implies $|\det(A) - \det(I)| = |\det(A) - 1| < \frac{1}{2}$. In particular, it implies $\det(A) > 0$. Because f' is continuous in $\overline{\mathbb{B}}^n$, we can define $C = \sup_{x\in\overline{\mathbb{B}}^n} ||f'(x) - I||_\infty < \infty$. For each $x \in \mathbb{B}^n$, we get:

$$\begin{aligned} ||f_t'(x) - I||_\infty &= ||(1 - t)f'(x) + (t - 1)I||_\infty \\ &= (1 - t)||f'(x) - I||_\infty \\ &\leq (1 - t)C. \end{aligned}$$

Thus, for t sufficiently close to 1 we get $(1-t)C < \xi$ and this implies $\det(f_t'(x)) > 0$ for all $x \in \mathbb{B}^n$.

Lastly, we prove $f_t(\mathbb{B}^n) = \mathbb{B}^n$ for t sufficiently close to 1. The two previous properties assure that $f_t(\mathbb{B}^n)$ is open for t close enough to 1, due to the open mapping theorem. Let $x \in \mathbb{B}^n$, then $||f_t(x)|| \leq t||f(x)|| + (1 - t)||x|| < 1$. This settles $f_t(\mathbb{B}^n) \subset \mathbb{B}^n$. For the reverse inclusion, suppose there exists $p_t \in \mathbb{B}^n \backslash f_t(\mathbb{B}^n)$. We have that $f_t(\overline{\mathbb{B}^n})$ is compact. Therefore, we can take $f_t(q_t) \in f_t(\overline{\mathbb{B}^n})$ that minimizes its distance to p_t. In particular,

$$||f(q_t) - p_t|| \leq ||f_t(p_t) - p_t|| = (1 - t)||f(p_t) - p_t||.$$

If $q_t \in \mathbb{B}^n$, then there exists an open ball V centered at $f_t(q_t) \in f_t(\mathbb{B}^n)$ such that $V \subset f_t(\mathbb{B}^n) \subset f_t(\overline{\mathbb{B}^n})$ and thus $f_t(q_t)$ cannot minimize the distance to p_t. Hence $q_t \in \mathbb{S}^{n-1}$ and, since f_t fixes $\mathbb{S}^{n-1}$, $f_t(q_t) = q_t$. Hence;

$$||q_t - p_t|| \leq (1-t)||f(p_t) - p_t||.$$

Recall that f has Lipschitz constant M in $\overline{\mathbb{B}}^n$. This allows to bound $||f(p_t)-p_t||$ as follows

$$\begin{aligned} ||f(p_t) - p_t|| &\leq ||f(p_t) - f(q_t)|| + ||q_t - p_t|| \\ &\leq (M+1)||q_t - p_t||. \end{aligned}$$

Finally

$$\begin{aligned} ||q_t - p_t|| &\leq (1-t)(M+1)||q_t - p_t|| \\ 1 &\leq (1-t)(M+1). \end{aligned}$$

For t close enough to 1 we arrive to a contradiction, settling $f_t(\mathbb{B}^n) = \mathbb{B}^n$. The aforementioned change of variables settles condition (iii). As argued, (i), (ii) and (iii) result in a contradiction, and thus there is no map f satisfying the conditions (1), (2) and (3).

□

As we will see, in the applications the maps involved are continuous but, eventually, not differentiable. Borsuk lemma can be extended to the most general case where, in the statement, the function is only continuous. For further details on this see Kannai (1981). To get an idea of how strong this lemma is, BFPT for the case $X = \overline{\mathbb{B}}^n$ is a direct implication.

Theorem 5. Let $f : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ be a continuous map. Then, there exists $x^* \in \overline{\mathbb{B}}^n$ such that $f(x^*) = x^*$.

Proof. Let us suppose by contradiction that the statement is false. Then, there exists $\phi : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ with no fixed point. Using ϕ let us define $r : \overline{\mathbb{B}}^n \to \mathbb{S}^{n-1}$ as the intersection point of the open ray that starts in $\phi(x)$ and passes through x, with $\mathbb{S}^{n-1}$. To define it explicitly as a formula we must find for each x a $t > 0$ such that

$$||\phi(x) + t(x - \phi(x))||^2 = 1,$$

which is equivalent to

$$||\phi(x)||^2 + 2t\langle\phi(x), x - \phi(x)\rangle + t^2||x - \phi(x)||^2 = 1.$$

Then this quadratic equation on t has only one positive solution which is given by

$$t(x) = -\frac{\langle\phi(x), x - \phi(x)\rangle}{||x - \phi(x)||^2} + \left[\frac{\sqrt{\langle\phi(x), x - \phi(x)\rangle^2 + ||x - \phi(x)||^2(1 - ||\phi(x)||^2)}}{||x - \phi(x)||^2}\right].$$

Thus, r must be

$$r(x) = \phi(x) + t(x)(x - \phi(x)).$$

Of course, t is continuous since it was defined explicitly, and thus r is too. In addition, r fixes $\mathbb{S}^{n-1}$ which can be easily seen using the geometric interpretation of r. Thus, r is a retraction from $\overline{\mathbb{B}}^n$ to $\mathbb{S}^{n-1}$, and a contradiction arises due to Lemma 3 in its most general statement. $\square$

Now we continue with the elements leading to the proof of the main result. The following propositions allow us to generalize Theorem 5 for X a non-empty compact convex subset.

Definition 6. Given a compact and convex set S in $\mathbb{R}^n$, denote the projection operator over S as $\pi_S : \mathbb{R}^n \to S$. It is implicitly defined by the relation $d(x, \pi_S(x)) = d(x, S)$, where

$$d(x, S) = \inf\{d(x, s) = ||x - s|| : s \in S\}.$$

Proposition 7. π_S is well defined.

Proof. Given a fixed $x \in \mathbb{R}^n$ there exists at least one s such that $d(x, s) = d(x, S)$ since S is compact. Now assume that there are two such points; let them be s_1 and s_2. Let h be the projection of x over the line that connects s_1 and s_2. It is not hard to see that h lies in between s_1 and s_2 since the triangle with vertices x, s_1 and s_2 is isosceles with $d(x, s_1) = d(x, s_2)$. Furthermore, h belongs to S due to its convexity. This is a contradiction since

$$d(x, h) < d(x, s_1) = d(x, S) \leq d(x, h).$$

Hence, there is a unique point that minimizes the distance between x and S. $\square$

Proposition 8. π_S is a continuous application.

Proof. Let p and q be two arbitrary different points in $\mathbb{R}^n$. Firstly we realise that

$$\langle \pi_S(p) - \pi_S(q), q - \pi_S(q) \rangle \leq 0.$$

Otherwise denoting $v = \pi_S(p) - \pi_S(q)$ and $w = q - \pi_S(q)$. Clearly, by the previous proposition and the contradiction assumption, $||v|| > 0$ and

$$d(\pi_S(q) + \varepsilon v, q)^2 = ||w||^2 + \varepsilon^2 ||v||^2 - 2\langle \varepsilon v, w \rangle$$
$$d(\pi_S(q) + \varepsilon v, q)^2 = ||w||^2 + \varepsilon ||v||^2 \left(\varepsilon - 2\frac{\langle v, w \rangle}{||v||^2} \right).$$

Then for a positive ε small enough,

$$d(\pi_S(q) + \varepsilon v, q) < ||w|| = d(q, \pi_S(q)) = d(q, S),$$

which is a contradiction since $\pi_S(q) + \varepsilon v$ belongs to S due to its convexity. Now with this last result we can conclude the following:

$$d(q, \pi_S(p))^2 = d(q, \pi_S(q))^2 + d(\pi_S(p), \pi_S(q))^2 - 2\langle \pi_S(p) - \pi_S(q), q - \pi_S(q) \rangle.$$
$$d(q, \pi_S(p))^2 \geq d(q, \pi_S(q))^2 + d(\pi_S(p), \pi_S(q))^2. \tag{1}$$

On the other hand, thanks to the triangular inequality,

$$d(p, q) + d(p, \pi_S(p)) \geq d(q, \pi_S(p)). \tag{2}$$

Then, with both (1) and (2) at our disposal we can conclude that

$$d(p, q)^2 + 2d(p, q)d(p, \pi_S(p)) + d(p, \pi_S(p))^2 \geq d(q, \pi_S(p))^2.$$
$$d(q, \pi_S(p))^2 \geq d(q, \pi_S(q))^2 + d(\pi_S(p), \pi_S(q))^2.$$
$$d(p, q)^2 + 2d(p, q)d(p, \pi_S(p)) + d(p, \pi_S(p))^2 \geq d(q, \pi_S(q))^2 + d(\pi_S(p), \pi_S(q))^2.$$
$$d(p, q)^2 + 2d(p, q)d(p, \pi_S(p)) + d(p, \pi_S(p))^2 - d(q, \pi_S(q))^2 \geq d(\pi_S(p), \pi_S(q))^2.$$

Since $d(p,q)^2$, $2d(p,q)d(p,\pi_S(p))$ and $d(p,\pi_S(p))^2 - d(q,\pi_S(q))^2$ get closer to 0 as q gets closer to p[1] we can conclude that $d(\pi_S(p),\pi_S(q))^2$ converges to 0 as q converges to p, proving that π_S is continuous. □

We give now some last additional results, essential to prove BFPT.

Proposition 9. Let S be a retract of Y. If Y possesses the fixed point property, then S does as well.

Proof. Let $f : S \to S$ be a continuous map and $r : Y \to S$ the retraction. First, $f \circ r : Y \to S \subset Y$ is continuous, and since Y possesses the fixed point property, there exists $y \in Y$ such that $y = f(r(y)) \in S$. However, since $r(y) = y$, we obtain $y = f(y)$, which proves S has the desired property. □

Proposition 10. Let $S \subset \overline{\mathbb{B}}^n_\delta$ be a closed and convex set (therefore compact), where

$$\overline{\mathbb{B}}^n_\delta = \{x \in \mathbb{R}^n : \ ||x|| \leq \delta\}.$$

Then, there exists a retraction $r : \overline{\mathbb{B}}^n_\delta \to S$.

Proof. Let $r : \overline{\mathbb{B}}^n_\delta \to S$ be the restriction over $\overline{\mathbb{B}}^n_\delta$ of the projection operator π_S. By proposition 8 it is straightforward to see that r is continuous. It is even clearer that r fixes S. □

Proposition 11. Let A and B be two homeomorphic spaces. If A possesses the fixed point property, then B does as well.

Proof. Let $f : B \to B$ be a continuous map and consider the homeomorphism $g : B \to A$. Then $g \circ f \circ g^{-1}$ is a continuous map from A to A. There exists therefore $a^* \in A$ such that $a^* = (g \circ f \circ g^{-1})(a^*)$. Apply g^{-1} on both sides to get $g^{-1}(a^*) = f(g^{-1}(a^*))$. Therefore $b^* = g^{-1}(a^*)$ is a fixed point for f. □

Hereafter the main proof of this document: Brouwer Fixed Point Theorem. As we shall see, it is a consequence of all previous propositions and theorems.

Proof. Since X is convex and compact, there exists $\delta > 0$ such that $X \subset \overline{\mathbb{B}}^n_\delta$. Using the map $\varphi(x) = x/\delta$, $\varphi : \overline{\mathbb{B}}^n_\delta \to \overline{\mathbb{B}}^n$, we can establish that $\overline{\mathbb{B}}^n_\delta$ and $\overline{\mathbb{B}}^n$

[1] This arises from the fact that $x \to d(x,S)$ is a continuous function.

are homeomorphic. By Theorem 5, we know that there is $x^* \in \overline{\mathbb{B}}^n$ such that $f(x^*) = x^*$, f continuous. Then, by Proposition 11, $\overline{\mathbb{B}}^n_\delta$ possesses the fixed point property. Finally, by Proposition 10, X is a retract of $\overline{\mathbb{B}}^n_\delta$ and, by Proposition 9, X possesses the fixed point property too. □

We have proven BFPT using the stronger version of Borsuk's lemma. This lemma was proven in a weaker case, as the proof of the general case involves more sophisticated tools. Besides this lemma, our proof is relatively simple. Hereafter we present applications in general equilibrium theory. We start by situating ourselves in an economic context and then prove a highly significant result in economic theory that makes use of Brouwer's fixed-point theorem: the existence of Walrasian equilibrium.

3 Existence of the Walrasian equilibrium

The purpose of this section is to illustrate the importance of Brouwer Fixed Point Theorem in general equilibrium theory. Even if the content might be standard in the literature, we derive some results on our own.

For the following definitions and standard framework we mainly follow Ellickson (1993), Mas-Colell et al. (1995), and Echenique (2023).

Let $i = 1, ..., I$ be the consumers of the economy, $X_i \subset \mathbb{R}^L_+$ their consumption sets, $\succeq_i$ their preferences over X_i and $\omega_i \in \mathbb{R}^L_+$ their endowment. Assume furthermore that the preferences $\succeq_i$ can be represented through utility functions $u^i(\cdot)$.

A pure exchange economy is

$$\mathcal{E} = (\omega^i, u^i)_{i=1}^I.$$

Definition 12. An allocation for the pure exchange economy is a collection of consumption vectors

$$x = (x_1, ..., x_I) \in \prod_{i=1}^I X_i \subset \mathbb{R}^{IL}_+$$

Hereafter we define the notion of Walrasian equilibrium for this economy following Echenique (2023).

Definition 13. Given a pure exchange economy, an allocation x^* and a price vector $p = (p_1, ..., p_L)$ constitute a Walrasian equilibrium if

1. $\forall\, i$, $x_i^*(p, p\cdot\omega_i) \in X_i$ is maximal with respect to the choice for $\succeq_i$ over the set
$$B = \Big\{x_i \in X_i : \ p\cdot x_i \leq p\cdot\omega_i\Big\}.$$
2. $\sum_i x_i^* = \sum_i \omega_i$.

In this framework, when preferences are rational and continuous, and $\sum_i \omega_i > 0$, an allocation x^* and a price vector $p = (p_1, ..., p_L)$ constitute a Walrasian equilibrium if

1. for each $i = 1, ..., I$, $x_i^* \in B(p, p\cdot\omega_i) = \{x_i \in \mathbb{R}_+^L : p\cdot x_i \leq p\cdot\omega_i\}$ and maximices $u_i(\cdot)$ over $B(p, p\cdot\omega_i)$.
2. $\sum_{i=1}^I x_i^*(p, p\cdot\omega_i) = \sum_{i=1}^I \omega_i$.

Remark. The difference with the previous statement is that , rationality and continuity of the preferences $\succeq_i$ imply the existence of a utility functions $u_i(\cdot)$.

Definition 14. We define the aggregated demand excess function by

$$z(p) = \sum_i z_i(p) = \sum_{i=1} \underbrace{[x_i^*(p, p\cdot\omega_i) - \omega_i]}_{\text{individual excess of demand}} \quad .$$

Since $p > 0$, $z : \mathbb{R}_{++}^L \to \mathbb{R}^L$.

Before addressing the main issue of this section, we present some properties of this function, very relevant in economic theory but also for the proof of the main result.

Lemma 15. Maximum principle. Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces and $f : \mathcal{X}\times\mathcal{Y} \to \mathbb{R}$ be a continuous function with respect the product topology over $\mathcal{X}\times\mathcal{Y}$, and let $\Gamma : Y \rightrightarrows X$ be a compact-valued correspondence (see Lucas et al. (1988)) s.t. $\Gamma(y) \neq \emptyset\ \forall\, y \in \mathcal{Y}$. Let us define the value function $f^* : \mathcal{Y} \to \mathbb{R}$:

$$f^*(y) = \sup\{f(x, y) : \ x \in \Gamma(y)\}$$

and the set of maximizers $\Gamma^* : \mathcal{Y} \to \mathcal{X}$:

$$\Gamma^*(y) = \text{argmax}\{f(x, y) : \ x \in \Gamma(y)\}.$$

If Γ is both upper and lower hemicontinuous at y, then f^* is continuous and Γ^* upper hemicontinuous, non empty and compact valued.

Lema 15 fits in the context of the utility maximization problem:

$$\begin{cases} \max & u(x) \\ \text{s.a.} & p \cdot x \leq I \\ & x \in \mathbb{R}^L_+. \end{cases}$$

Indeed, set $X = \mathbb{R}^L_+$ the space of commodities, $\mathcal{Y} = \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ the space of prices: (p, I), $f(x, y) = u(x)$ the utility function and

$$\Gamma(y) = B(p, I) = \{x \geq 0 : \; p \cdot x \leq I\}$$

the consumer budget set. Then:

1. The indirect utility function $v(p, I)$ is continuous.

2. Marshallians demands $x^*(p, I)$ are continuous.

Proposition 16. If $(\succeq_i, \omega_i)_{i=1}^I$ is a pure exchange economy s.t. $\overline{\omega} = \sum_{i=1}^I \omega_i \gg \mathbf{0}$ and $\succeq_i$ is continuous, strictly convex and strictly monotonic, then $z(\cdot)$ satisfies the following properties:

1. z is continuous.

2. z is homogeneous of degree zero: $z(\lambda p) = z(p) \; \forall \; \lambda > 0$.

3. z satisfies Walras law: $\forall \; p \in \mathbb{R}^L_{++} : \; p \cdot z(p) = 0$.

4. $\exists \; M > 0$ such that $\forall \; \ell = 1, ..., L$ and $p \in \mathbb{R}^L_{++}$: $z_\ell(p) > -M$.

Proof. Item by item:

1. The continuity of $u^i(\cdot)$ and properties of $B(p, I)$ ensures by Lemma 15 the continuity of $x_i^*(p, p \cdot \omega_i)$ for all $i = 1, ..., I$, and therefore the continuity of z.

2. For each consumer $i = 1, ..., I$ its budget set

$$B(p, p \cdot \omega_i) = \{x_i \geq 0 : p \cdot x_i \leq p \cdot \omega_i\}$$

clearly remains unchanged if $p \to \lambda p$.

3. Since $\succeq_i$ is strictly monotonic for each consumer,

$$\forall\ i = 1, ..., I: \ \underbrace{p \cdot x_i^*(p, p \cdot \omega_i)}_{\text{expenditure}} = \underbrace{p \cdot \omega_i}_{\text{income from endowment}}.$$

Hence:

$$\sum_{i=1}^{I} p \cdot x_i^*(p, p \cdot \omega_i) = \sum_{i=1}^{I} p \cdot \omega_i$$
$$p \cdot \left(\sum_{i=1}^{I} x_i^*(p, p \cdot \omega_i) - \omega_i \right) = 0$$
$$p \cdot z(p) = 0.$$

4. Since $x_{\ell i}^*(p, p \cdot \omega_i)$ is positive for each consumer $i = 1, ..., I$ and good $\ell = 1, ..., L$:

$$z_\ell(p) \geq -\overline{\omega}_\ell.$$

Let $M > \max_{\ell=1,...,L}\{\overline{\omega}_\ell\}$. Then, $z_\ell(p) > -M$ for all ℓ and $p \in \mathbb{R}^L_{++}$.

□

Another property that $z(\cdot)$ satisfies is that if $\{p_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^L_{++}$ converges to $\overline{p} \neq 0$ such that there exists $\ell: \ \overline{p}_\ell = 0$, then

$$\max\{z_1(p_n), ..., z_L(p_n)\} \to \infty.$$

The proof can be found in Echenique (2023). We now focus on the main topic of this section: how Brouwer Fixed Point Theorem is applied in order to proof Walrasian equilibrium existence.

Theorem 17. Existence of Walrasian equilibrium. In the context of Proposition 16, for $z : \mathbb{R}^L_+ \to \mathbb{R}^L$, there exists $p^* \in \mathbb{R}^L_+$ such that $z(p^*) \leq 0$. Furthermore, if $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$, there exists p^* such that $z(p^*) = 0$.

The proof of this theorem relies on Theorem 1. Nowadays the following is well known and can be found (following similar or very different approaches) in, for example, Mas-Colell et al. (1995), Varian (1992) or Ellickson (1993).

Proof. First, since z is homogeneous of degree zero, we can restrict p to the Δ

(also known as $n-$dimensional simplex), defined as follows:

$$\Delta = \left\{ p \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell = 1 \right\}.$$

This set is clearly convex and compact. Indeed, given $p_1, p_2 \in \Delta$ and $\theta \in [0,1]$,

$$p_3 = \theta p_1 + (1-\theta) p_2 \in \Delta :$$

$$\begin{aligned}\sum_{\ell=1}^{L} p^3_\ell &= \sum_{\ell=1}^{L} \theta p^1_\ell + (1-\theta) p^2_\ell \\ &= \theta \sum_{\ell=1}^{L} p^1_\ell + (1-\theta) \sum_{\ell=1}^{L} p^2_\ell \\ &= \theta + (1-\theta) = 1.\end{aligned}$$

With respect to the compactness, Δ is closed since it is the intersection of the orthant $\mathbb{R}^L_+$ and the hyperplane $H\left((1,...,1),1\right)$. It is bounded since $\Delta \subset [0,1]^L$. Hence, since all of this occurs in $\mathbb{R}^L$, Δ is a compact set. It is therefore possible to apply Brouwer fixed point over Δ. We would only need to prove that $z(p) + p$ maps Δ onto Δ. However, this is not the case in general. This is where the following trick is employed, which allows us to conclude the matter using Brouwer's Fixed Point Theorem. Let us define $\Psi : \Delta \to \mathbb{R}^L$ defined as follows:

$$\Psi_\ell = \frac{p_\ell + \max\{0, z_\ell(p)\}}{1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(p)\}}, \;\; \forall \; \ell = 1, ..., L.$$

Since $\sum_{\ell=1}^{L} p_\ell = 1$,

$$\sum_{\ell=1}^{L} \Psi_\ell = \sum_{\ell=1}^{L} \left\{ \frac{p_\ell + \max\{0, z_\ell(p)\}}{1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(p)\}} \right\} = 1,$$

i.e., $\Psi(\Delta) \subset \Delta$. Hence, by Theorem 1, there exists p^* such that $\Psi(p^*) = p^*$. This yields to: $\forall \; \ell = 1, ..., L$

$$p^*_\ell = \frac{p^*_\ell + \max\{0, z_\ell(p^*)\}}{1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(p^*)\}}$$

$$p^*_\ell \left(1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(p^*)\} \right) = p^*_\ell + \max\{0, z_\ell(p^*)\}$$

$$p_\ell^* \sum_{\ell=1}^{L} \max\{0, z_\ell(p^*)\} = \max\{0, z_\ell(p^*)\}$$

$$z_\ell(p^*) p_\ell^* \sum_{\ell=1}^{L} \max\{0, z_\ell(p^*)\} = z_\ell(p^*) \max\{0, z_\ell(p^*)\}$$

$$\underbrace{\sum_{\ell=1}^{L} z_\ell(p^*) p_\ell^*}_{=0} \left[\sum_{\ell=1}^{L} \max\{0, z_\ell(p^*)\}\right] = \sum_{\ell=1}^{L} z_\ell(p^*) \max\{0, z_\ell(p^*)\}$$

Therefore,

$$\sum_{\ell=1}^{L} z_\ell(p^*) \max\{0, z_\ell(p^*)\} = 0. \tag{3}$$

Equation 3 points out that $z_\ell(p^*) \leq 0, \ \forall \ \ell = 1, ..., L$. Finally, once again by Walras Law (Proposition 16) since we must have

$$\sum_{\ell=1}^{L} p_\ell^* z_\ell(p^*) = 0 \tag{4}$$

with $p_\ell \geq 0$, combining (4) with Equation 3 we must have $p_\ell z_\ell(p^*) = 0$ for all $\ell = 1, ..., L$. Finally, for $p_\ell > 0$, necessarily $z_\ell(p^*) = 0$ for all $\ell = 1, ..., L$, which concludes the proof. □

Theorem 17 allows us to appreciate the power of Brouwer's fixed-point argument: it has been proven under very reasonable assumptions about consumer preferences that there exists a price vector that clears the market. The applications of general equilibrium theory are vast, as mentioned in Echenique and Wierman (2012). One of the main challenges is for instance to compute this equilibrium, which is of great interest in macroeconomics. To conclude this work, by way of an example, we will compute the vector of prices for Walrasian equilibrium when all consumers share the same preferences: Cobb-Douglas, with $I = L - 1$.

Example 18. Let us consider a pure exchange economy where all consumers have the same preferences: Cobb-Douglas

$$u^i(x_{1i}, ..., x_{Li}) = \prod_{i=1}^{L} x_{\ell i}^{\alpha_{\ell i}},$$

such that $\sum_{\ell=1}^{L} \alpha_{\ell i} = 1$ and $\alpha_{\ell i} \in (0,1)$. Let us denote, as usual, $\omega_{\ell i}$ the endowment of ℓ good of consumer i, such that $\overline{\omega} > 0$. The maximization problem faced by all consumers is

$$\mathcal{P}_i \begin{cases} \max & u^i(x_i) = \prod_{\ell=1}^{L} x_{\ell}^{\alpha_{\ell i}} \\ \text{s.a.} & \sum_{\ell=1}^{L} p_\ell x_{\ell i} = \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \\ & x_{\ell i} \geq 0. \end{cases}$$

The Cobb-Douglas function satisfies Inada conditions. Therefore, the solution is not a corner solution Chávez and Gallardo (2024) and we can reduce the restrictions to $x_{\ell i} > 0$. From this last, and using that $\ln(\cdot)$ is strictly increasing and concave, we might re-write $\mathcal{P}_i$ as follows:

$$\mathcal{P}_i \begin{cases} \max & \ln[u^i(x_i)] = \ln\left(\prod_{\ell=1}^{L} x_{\ell i}^{\alpha_{\ell i}}\right) = \sum_{\ell=1}^{L} \alpha_{\ell i} \ln x_{\ell i} \\ \text{s.a.} & \sum_{\ell=1}^{L} p_\ell x_{\ell i} = \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \\ & x_{\ell i} > 0. \end{cases}$$

$\mathcal{P}_i$ can be solved using Lagrange Chávez and Gallardo (2024)

$$\mathscr{L}(\{x_{\ell i}\}_{\ell=1}^{L}, \lambda) = \sum_{\ell=1}^{L} \alpha_{\ell i} \ln x_{\ell i} + \lambda \left[\sum_{\ell=1}^{L} p_\ell \omega_{\ell i} - \sum_{\ell=1}^{L} p_\ell x_{\ell i}\right].$$

First-order conditions will be enough to characterize the equilibrium in reason of the differentiability and strict concavity of the utility function:

$$\forall\ \ell = 1, ..., L: \ \frac{\partial \mathscr{L}}{\partial x_{\ell i}} = \frac{\alpha_{\ell i}}{x_{\ell i}} - \lambda p_\ell = 0.$$

Summing over ℓ

$$\underbrace{\sum_{\ell=1}^{L} \alpha_{\ell i}}_{=1} = \sum_{\ell=1}^{L} \lambda p_\ell x_{\ell i} \implies \lambda = \frac{1}{\sum_{\ell=1}^{L} p_\ell x_{\ell i}} = \frac{1}{\sum_{\ell=1}^{L} p_\ell \omega_{\ell i j}}.$$

From this,

$$\forall\ i, \ell: \ x_{\ell i} = \frac{\alpha_{\ell i}}{p_\ell}\left(\sum_{\ell=1}^{L} p_\ell \omega_{\ell i}\right).$$

Since the clearing marker conditions impose

$$\sum_{i=1}^{I} x_{\ell i} = \sum_{i=1}^{I} \omega_{\ell i}, \ \forall\ \ell = 1, ..., L$$

we have

$$\sum_{i=1}^{I} \alpha_{\ell i} \left(\sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \right) = p_\ell \sum_{i=1}^{I} \omega_{\ell i}, \ \forall \ \ell = 1, ..., L. \tag{5}$$

After normalizing without loss of generality $p_1 = 1$, the right side of Equation 5 can be written as follows in a compact way

$$\underbrace{\begin{bmatrix} \sum_{i=1}^{I} \omega_{1i} & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{I} \omega_{2i} & & \vdots \\ \vdots & & \ddots & \\ 0 & & & \sum_{i=1}^{I} \omega_{Li} \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} \underbrace{p_1}_{=1} \\ p_2 \\ \vdots \\ p_L \end{bmatrix}}_{\tilde{p}}.$$

On the other hand, the left side of Equation 5 can be put under matrix form too as follows (writing $\omega_{i\ell}$ and $\alpha_{i\ell}$)

$$\underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1L} \\ \alpha_{21} & \alpha_{22} & & \vdots \\ \vdots & & \ddots & \\ \alpha_{I1} & & & \alpha_{IL} \end{bmatrix}^T \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1L} \\ \omega_{21} & \omega_{22} & & \vdots \\ \vdots & & \ddots & \\ \omega_{I1} & & & \omega_{IL} \end{bmatrix}}_{= \underbrace{\begin{bmatrix} \sum_{i=1}^{I} \alpha_{i1}\omega_{i1} & \sum_{i=1}^{I} \alpha_{i1}\omega_{i2} & \cdots & \sum_{i=1}^{I} \alpha_{i1}\omega_{iL} \\ \sum_{i=1}^{N} \alpha_{i2}\omega_{i1} & \ddots & & \vdots \\ \vdots & & \ddots & \\ \sum_{i=1}^{I} \alpha_{iL}\omega_{i1} & & & \sum_{j=1}^{L} \alpha_{iL}\omega_{iL} \end{bmatrix}}_{=\Sigma}} \begin{bmatrix} \underbrace{p_1}_{=1} \\ p_2 \\ \vdots \\ p_L \end{bmatrix}$$

If we denote b as the first column vector of matrix Σ, excluding the first entry, A as the sub-matrix $I \times (L-1)$, which includes all columns except the first column of Σ, $\overline{p}$ as the truncated price vector, excluding $p_1 = 1$, and $\overline{W}$ as the $(L-1) \times (L-1)$ matrix (excluding the first row and first column of W), then:

$$\overline{p} = (\overline{W} - A^{-1}b, \ p_1 = 1.$$

It is easy to check that the conditions of Theorem 17 are satisfied, backing up our conclusion.

4 Conclusion

In this document, we have provided a proof of Borsuk's lemma for continuously differentiable retractions, as outlined in Laczkovich and Sos (2017) (which provides a sketch of the general argument). Theorem 5 follows as a direct result. We then presented and proved several results that allowed us to prove the general case of Brouwer Fixed Point Theorem. We followed the statements according to Ok (2007), providing our own constructions

After completing the proof of the BFPT, we delved into General Equilibrium Theory. The Walrasian existence theorem for pure exchange economies was proven, and we also presented an example which is not of standard character, illustrating the power of Theorem 17.

We hope the reader will find this document highly interesting and useful, especially for better understanding and applying how Brouwer's Fixed Point Theorem, after being proven in a slightly more restrictive case, is utilized in general equilibrium to derive one of the main results of the theory.

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