

Pontificia Universidad Católica del Perú

Economics Major

November 24, 2024

Recitation 8 ECO 263

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Stochastic dominance

Exercise 1 (Mas-Colell et al.). Let $F(\cdot)$ and $G(\cdot)$ be lotteries defined over an amount of money x , and let $U : L \rightarrow \mathbb{R}$ be the expected utility function in the von Neumann-Morgenstern (vNM) framework, with a utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ that is non-decreasing.

1. Show that if $G(x) \geq F(x) \forall x$, then $U(F) \geq U(G)$ (First-Order Stochastic Dominance, FSD).
2. If, in addition, the utility function is strictly concave and the lotteries offer the same expected payoff, show that if

$$\int_0^x G(t) dt \geq \int_0^x F(t) dt \quad \forall x,$$

then $U(F) \geq U(G)$ (Second-Order Stochastic Dominance, SSD).

Exercise 2 (T. Sarver, Duke). Suppose that F is the uniform distribution on the interval $[0, a]$:

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x/a, & \text{if } 0 \leq x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

Similarly, suppose that G is the uniform distribution on the interval $[0, b]$. If $a \geq b$, show that $F \geq_{FOSD} G$.

Adverse selection

Exercise 3. Consider the following market for used cars, similar to Akerlof's, where the quality of these cars is given by $x \in [0, 1]$. A car of quality x is valued at x by the buyer and $v(x)$ by the seller, where $v(\cdot)$ is a continuous, strictly increasing function such that $v(x) \leq x$ for all x . If the density of qualities is $f(x)$, determine the equilibrium under the following conditions:

- Buyers and sellers know the quality of each car.
- Neither buyers nor sellers know the quality of each car.
- If f is a uniform distribution on $[0, 1]$ and $v(x) = x^2$, and only the sellers know the quality of the car.
- Determine the welfare loss caused by the information asymmetry.

Moral hazard

Exercise 4. Prove that, if effort is not observable but the manager is risk-neutral, the **payment scheme** generates the same expected utility and profit.

Exercise 5. Suppose a principal-agent relationship where two outcomes are possible, valued at 50,000 and 25,000 dollars. The agent must choose between two possible effort levels. The probability distribution over outcomes as a function of effort levels is given in the following table:

	25,000	50,000
e_1	1/4	3/4
e_2	1/2	1/2

Assume the principal is risk-neutral and the agent is risk-averse, with preferences described by the following functions:

$$B(x, w) = x - w,$$

$$U(w, e) = 2\sqrt{w} - g(e),$$

where $g(e_1) = 40$ and $g(e_2) = 20$. The reservation utility level is $\bar{u} = 120$.

- Write the optimal contracts under symmetric information for each effort level and the profits obtained by the principal in each case. Which effort level does the principal prefer?
- Write the optimal contracts when there is a moral hazard problem. What is the effort level and the contract chosen by the principal? Where does the moral hazard problem manifest?

Exercise 6 (I. Segal, ECON 206 UCB). Consider a standard moral-hazard problem with the following features:

- The principal (p) and the agent (a) are both risk neutral. Let x be the verifiable output, e the agent's unobserved effort, and $w(x)$ the payment to the agent. The two parties' final utility levels are given by:

$$u_p = x - w(x), \quad u_a = w(x) - v(e),$$

where $v(\cdot)$ is a strictly increasing function of effort.

- The agent has finite wealth, which constrains the principal to offer incentive schemes $w(x) > 0$ for all x . This constraint guarantees that the agent is willing to work for the principal (no additional participation constraint is needed).
- Output can take three values: $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. Effort can take two values: $e_0 = 0$ and $e_1 = 1$. Normalize $v(0) = 0$.
- The probability of x given e , denoted $\pi(x | e)$, satisfies the Monotone Likelihood Ratio Property:

$$\frac{\pi(x_j | e = 1)}{\pi(x_j | e = 0)} > \frac{\pi(x_{j-1} | e = 1)}{\pi(x_{j-1} | e = 0)} \quad \text{for } j = 2, 3.$$

- Assume that a second-best solution to this problem induces the agent to choose e_1 . Show that in such a solution $w(x_1) = w(x_2) = 0$, and $w(x_3) > 0$.
- Assume now that the slope of the incentive scheme is restricted to lie between 0 and 1, i.e.,

$$\frac{w(x_j) - w(x_{j-1})}{x_j - x_{j-1}} \in [0, 1] \quad \text{for } j = 2, 3.$$

(This assumption can be rationalized by arguing that the agent could otherwise engage in arbitrage using either free disposal or by using external funds to boost revenues.) Assume again that a second-best solution induces the agent to choose e_1 . Show that in such a solution $w(x_1) = 0$, and that if $w(x_2) > 0$, then $w(x_3) - w(x_2) = 1$.

Suggested exercises

Stochastic Dominance

Exercise 7. Consider two random variables X and Y defined on a probability space $(\Omega, \mathcal{F}, P)$.

- Prove that if $X \geq Y$, then $F_X \geq_{FOSD} F_Y$. **Hint:** note that

$$\{\omega \in \Omega : X(\omega) \leq x\} \subset \{\omega \in \Omega : Y(\omega) \leq x\}.$$

- Show that the converse of the previous statement is not true. **Hint:** consider $\Omega = [0, 1]$, P as the uniform distribution, and $X(\omega) = 1 + \omega$, $Y = 2 - 2\omega$.

Exercise 8. Let X and Y be real-valued random variables, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. Discuss the truth of the following statements:

- a) If X stochastically dominates Y in the first order, then $g(X)$ stochastically dominates $g(Y)$ in the first order.
- b) If X stochastically dominates Y in the second order, then $g(X)$ stochastically dominates $g(Y)$ in the second order.

Adverse selection

Exercise 9. Consider a car market where the maximum available quality is $Q = 1.9$, and the distribution of q is uniform with $a = 0$ and $b = Q$. A car of quality q is valued at most by the buyer as q and by the seller as q/Q . It is assumed that there are sufficiently many buyers to drive the gains from trade on the seller's side. Since $q/Q < q$, the buyer assigns a higher value to the car than the seller, allowing both parties to trade the car at a price p between q/Q and q , generating a profit for the seller and a surplus for the buyer. In particular, if a car of quality q is traded at price p , the buyer obtains a utility

$$u(p, q) = q - p,$$

while the seller obtains a profit of

$$\pi(p, q, Q) = p - \frac{q}{Q}.$$

- a) Calculate the buyer's expected value assuming there is asymmetric information.
- b) Based on the previous part, calculate the cutoff point for cars offered by the seller, the associated price, and their profits.
- c) Calculate the valuation of cars by the buyer and seller under conditions of perfect information.

- d) If a buyer anticipates the interval where cars will not be offered, determine the new expected value of offered cars and the seller's decision. What happens to the new prices and the seller's profits?

Exercise 10. Consider a competitive labor market with many firms seeking to hire a worker for a specific position. The worker (seller of labor services) privately observes their own productivity θ , but firms (buyers of labor) cannot observe it. Firms offer a wage based on the worker's expected productivity:

$$\mathbb{E}[\theta] = \frac{1}{2}, \theta \in U[0, 1].$$

At this wage level, only workers with a productivity $\theta < 1/2$ would be interested in accepting the position, while those with $\theta > 1/2$ will remain unemployed. It is assumed that the cost of not working for the firm is working at home, where they receive a reservation wage of $1/2$.

- a) What wage will a rational employer offer?
- b) Who will accept the job at that wage level?

Moral Hazard

Exercise 11. Suppose Manuel wants to hire a research assistant. It is known that the probability of the assistant doing good or bad work ($x_i, i = 1, 2$) depends on the effort they put in, meaning everything is expressed as $\mathbb{P}(x | e)$. Two levels of effort are identified: high effort e_H and low effort e_L . On the other hand, Manuel is risk-neutral, so he maximizes the expected value of profits:

$$\pi = x_i - w_i,$$

where w_i is the salary that Manuel pays the research assistant, who is risk-averse. Therefore, the assistant's preferences are represented by:

$$U(w, e) = v(w) - g(e),$$

where v is strictly concave and increasing, and g is strictly convex and also increasing. Based on this, assuming the high effort level executed by the agent maximizes the principal's expected profits:

- a) Formulate and solve the principal's optimization problem under symmetric information. What will be the payment scheme w_i ?
- b) Formulate and solve the principal's optimization problem under asymmetric information. Explain the relationship between the salary and a higher value of $\mathbb{P}(x | e = e_H)$.

Exercise 12. Consider the following Principal-Agent problem, where the Agent chooses between two actions, $a \in \{a_1, a_2\} = \{0, 1\}$. The Principal pays the Agent a wage w_s , with $s \in \{1, 2, 3, 4\}$, contingent on observing the production $Y \in \{y_1, y_2, y_3, y_4\}$, where $y_1 < y_2 < y_3 < y_4$. The probability distribution of each level of production is as follows, $p_s(a_i) := \Pr\{Y = y_s \mid a = a_i\}$, $i \in \{1, 2\}$:

a_i	$p_1(a_i)$	$p_2(a_i)$	$p_3(a_i)$	$p_4(a_i)$
a_1	3/8	3/8	1/8	1/8
a_2	1/4	1/4	1/4	1/4

The Agent's utility function is:

$$U(w, a) = \sqrt{w} - a$$

Assume that the Agent's reservation utility is $u = 2$. The Principal is risk-neutral.

- If the action were observable, characterize the optimal contract to incentivize $a_2 = 1$.
- Suppose the action a_i is not observable. Characterize the optimal contract that implements a_2 . Use the following rule to determine salaries: $LR_k = \frac{p_k(a_2)}{p_k(a_1)}$, $w_s > (=) w_r \iff LR_s > (=) LR_r$. LR denotes likelihood ratio.
- Indicate the efficiency loss due to information asymmetry.

1 Moral Hazard: continuous effort

Exercise 13 (Mas-Colell et al.). Consider a hidden-action model where the principal is risk-neutral, while the manager's preferences are determined by:

$$UE(w, e \mid \phi) = \mathbb{E}[w] - \phi \text{Var}(w) - g(e),$$

with $g'(0) = 0$, $g^{(k)}(\cdot) > 0$ for $k = 1, 2, 3$, $e > 0$, and $\lim_{e \rightarrow \infty} g'(e) = \infty$. We consider $e \in \mathbb{R}_+$. The benefits subject to effort are distributed according to $\mathcal{N}(e, \sigma^2)$.

- If the payment scheme is linear $w(\pi) = \alpha + \beta\pi$, prove that the manager's expected utility given $w(\pi)$, e , and σ^2 is:

$$\alpha + \beta e - \phi \beta^2 \sigma^2 - g(e).$$

- Derive the optimal contract when e is observable.
- Derive the optimal linear payment scheme when effort is not observable. Analyze the effects of β and σ^2 .

Exercise 14. Suppose an employer (E) hires a lawyer (A) for representation. For an agent of type θ , the time required to produce legal services in the amount x is denoted as a , where $a = \frac{x}{2\theta}$. The agent can either be of low productivity (θ_1) or high

productivity (θ_2), with $\theta_2 > \theta_1 > 0$. Let y be the payment made to the lawyer. The agent has a reservation utility of 0, and their utility function is $U(y, a) = \sqrt{y} - a$. The principal E is risk-neutral, and legal services have a unit price of 1 monetary unit.

- a) If you observe the type of lawyer you are hiring, what contract would you offer? Is this contract efficient? Graph the equilibrium, including the optimal quantities, indifference curves, and isoprofit lines, with y on the vertical axis and x on the horizontal axis.
- b) Suppose now that you believe $\pi_1 = \Pr(\theta = \theta_1)$, $\pi_2 = \Pr(\theta = \theta_2)$, and assume $\theta_1 \geq (1 - \pi_1)\theta_2$. What would the set of contracts you offer look like now? Graph the equilibria, clearly indicating the optimal quantities, indifference curves, and isoprofit lines.

2 Advanced exercises

Exercise 15 (Mas-Colell et al.). Suppose the reservation price, opportunity cost of working, or production at home $r(\theta)$ is a continuous and strictly increasing function. Additionally, assume there exists a productivity value $\hat{\theta} \in [\underline{\theta}, \bar{\theta}]$ such that $r(\theta) > \theta$ for $\theta > \hat{\theta}$ and $r(\theta) < \theta$ for $\theta < \hat{\theta}$. The density of workers of type θ is $f(\theta)$ with $f(\theta) > 0$, $\forall \theta \in [\underline{\theta}, \bar{\theta}]$. Based on this, answer the following:

- a) Find the Pareto-efficient allocation. What will be the aggregate production? Graph $r(\theta)$ and θ .
- b) Find the competitive equilibrium in the labor market under information asymmetry, and prove that it is not Pareto efficient. Analyze it as a function of $\hat{\theta}$.
- c) Suppose $r(\theta) = \frac{1}{\theta}$, $\theta \sim U[1/2, 2]$, $r(\underline{\theta}) < \underline{\theta}$, and $r(\bar{\theta}) < \bar{\theta}$. Find the socially efficient labor allocation and the corresponding aggregate production.
- d) In the context of information asymmetry, find the conditional expectation of productivity.

Exercise 16 (I. Segal, ECON 206 UCB). An entrepreneur (E) has a project requiring an initial investment of k . The project's random output, $\tilde{x} > 0$, depends on E's choice of effort as follows: $\tilde{x} = e\tilde{y}$, where $e > 0$ is E's choice of effort and $\tilde{y}$ is a random variable distributed uniformly on $[0, 2]$. E's private cost of effort is $g(e) = \frac{1}{2}e^2$, and this effort is unobservable, while the output is observable and verifiable. E makes a take-it-or-leave-it offer to an investor (I). Assume that E has no starting capital, so I must pay the start-up cost k . The contract specifies the sharing rule of output such that when output is x , E keeps $w(x)$ for himself and the rest goes to I. E has limited liability, which constrains $w(x) > 0$ for all x . Both parties are risk-neutral, and the market interest rate is normalized to zero.

- a) Suppose that E's effort is observable and verifiable. Solve for the first-best level of effort. For what values of k would the project be worth undertaking if there were no moral hazard?

- b) Assume that the project is worth undertaking. Show that despite the unobservability of effort by E, an optimal contract implements the first-best level of effort.
Hint: Consider contracts of the form

$$w(x) = \begin{cases} 0 & \text{if } x < a, \\ w & \text{if } x \geq a. \end{cases}$$

- c) Now suppose that before output is observed by outsiders (including the principal), E can destroy or borrow output at no cost. What restrictions does this impose on the contract? Solve for the form of the optimal contract in this case.
Hint: You can use integration by parts to express the objective function and constraints through $w(0)$ and $w'()$.
- d) Interpret the resulting contract in financial terms.