Pontificia Universidad Católica del Perú
Economics Major

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Recitation 7
ECO 263

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Important: from the mathematical preliminaries, you must be clear with concepts from you course Statistical Inference (the ones that you must define). With respect to the implicit function theorem, you only need to understand where it is applied and the general idea.
Order: exercises 1-4 are mandatory. Exercises 5-7 and 12 are highly recommended. Exercises 8-11 are postgraduate level. Advanced exercises are from PhD and require some tools that are no delivered in Mathematics for Economists I-IV.
Suggestion: be efficient and start solving the mandatory exercises as well as the recommended. Pass to the others only if you have finished the others, or if you think that you can easily solve them.

Mathematical preliminaries

Probability theory

Define what is

1. A probability space. $(\Omega, \mathcal{F}, \mathbb{P})$
2. A random variable. $X : \Omega \to \mathbb{R}$.
3. A probability measure. $\mathbb{P} : \mathcal{F} \to [0, 1]$.
4. CDF, density, expectation and variance. $F_X, f_X, \mathbb{E}[X], \text{Var}(X)$.

Implicit function theorem

1. Used in: comparative statics (Micro 1, Macro 1). Now, see Exercise 13.

2. **Formal and general statement:** let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a C^1 function. Denote $(x, y) = (x_1, \cdots, x_n, y_1, \cdots, y_m) \in \mathbb{R}^{n+m}$. Fix $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(a, b) = 0$. If $J_{f,y}(a, b) = \left[\frac{\partial f_i}{\partial y_j}(a, b)\right]$ is invertible, then there exists $U \subset \mathbb{R}^n$, open, containing a, such that there exists a unique function $g : U \to \mathbb{R}^m$ with $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in U$. Moreover, g is C^1 and

$$\left[\frac{\partial g_i}{\partial x_j}(x)\right]_{m\times n} = -[J_{f,y}(x, g(x))]^{-1}_{m\times m}[J_{f,x}(x, g(x))]_{m\times n}.$$

3. **Interpretation:** if you have an equation $F(\theta^*, x^*) = 0$, where θ are parameters, and the conditions of the theorem are satisfied, then, we can write $x = x(\theta)$ for θ close to θ^*. The function $x(\theta)$ is C^1 and we can calculate $\frac{\partial x_i}{\partial \theta_j}$. Usually, $F = \nabla f$, so $F = 0$ corresponds to first order conditions $\nabla f = 0$.

Example 1. Consider $f(x, y) = x^2 + y^2$. The equation $f(x, y) = 1$ corresponds to the unity circle. A way to represent the unit circle is taking $y = \pm\sqrt{1 - x^2}$. It is only possible to do this *by parts*.

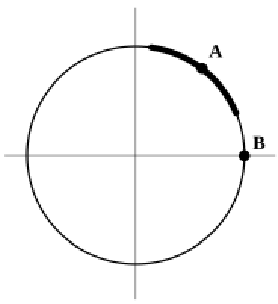

Figure 1: Fig1

Example 2. Given the utility maximization problem[1], assuming Inada conditions hold for the utility function $u(x, y)$, the problem is written as

$$\begin{aligned} &\max u(x, y) \\ &\text{s.t.} : p_x x + p_y y = I. \end{aligned}$$

We wish to determine $\frac{dx}{dI}$ and $\frac{dx}{dp_x}$. For simplicity, we use the notation

$$\frac{\partial u}{\partial x} = u_x, \text{ and } \frac{\partial u}{\partial y} = u_y.$$

[1]In the following chapter, we delve into optimization topics.

From the first-order conditions, we obtain the following system:

$$\frac{\partial \mathcal{L}}{\partial x} = u_x - \lambda p_x = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = u_y - \lambda p_y = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0.$$

Taking differentials in these three equations, we get

$$\begin{aligned} d(u_x - \lambda p_x) &= u_{xx}dx + u_{xy}dy - d\lambda p_x - \lambda dp_x = 0, \\ d(u_y - \lambda p_y) &= u_{yy}dy + u_{yx}dx - d\lambda p_y - \lambda dp_y = 0, \\ d(I - p_x x - p_y y) &= dI - dxp_x - xdp_x - dyp_y - ydp_y = 0. \end{aligned}$$

The Inada conditions ensure that the goods are normal and preferences are convex. Therefore:

$$u_{xy} > 0, \ u_{xx}, u_{yy} < 0.$$

Thus, the system of equations becomes

$$\begin{pmatrix} u_{xx} & u_{xy} & -p_x \\ u_{yx} & u_{yy} & -p_y \\ -p_x & -p_y & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ d\lambda \end{pmatrix} = \begin{pmatrix} \lambda dp_x \\ \lambda dp_y \\ xdp_x + ydp_y - dI \end{pmatrix}.$$

Applying Cramer's rule to obtain

$$\frac{dx}{dI},$$

considering $dp_x = dp_y = 0$ (keeping prices constant), we compute

$$dx = \frac{1}{|A|} \begin{vmatrix} 0 & u_{xy} & -p_x \\ 0 & u_{yy} & -p_y \\ -dI & -p_y & 0 \end{vmatrix},$$

where

$$\begin{aligned} |A| &= \det \begin{pmatrix} u_{xx} & u_{xy} & -p_x \\ u_{yx} & u_{yy} & -p_y \\ -p_x & -p_y & 0 \end{pmatrix} \\ &= \underbrace{-u_{xx}p_y^2 + u_{xy}p_x p_y + u_{yx}p_x p_y - u_{yy}p_x^2 > 0.}_{\text{convex preferences, i.e., } u \text{ is quasi-concave}} \end{aligned}$$

Then,

$$\frac{dx}{dI} = \frac{u_{xy}p_y - p_x u_{yy}}{|A|} > 0.$$

The inequality follows from the fact that marginal utility decreases for each good

$u_{xx}, u_{yy} < 0$, while the cross derivatives are positive due to convex preferences $u_{xy}, u_{yx} > 0$.

Regarding $\frac{dx}{dp_x}$, considering $dp_y = dI = 0$

$$dx = \frac{1}{|A|} \begin{vmatrix} \lambda dp_x & u_{xy} & -p_x \\ 0 & u_{yy} & -p_y \\ xdp_x & -p_y & 0 \end{vmatrix}.$$

Expanding the determinant in the numerator,

$$\begin{aligned} \frac{dx}{dp_x} &= \frac{-\lambda p_y^2 - xp_y u_{xy} + xp_x u_{xx}}{|A|} \\ &= \frac{-\lambda p_y^2 + x(u_{yy}p_x - u_{xy}p_y)}{|A|} \\ &= -\frac{\lambda p_y^2}{|A|} - x\frac{dx}{dI} < 0. \end{aligned}$$

This equation indicates that when the price of a good increases, demand for that good decreases due to a substitution effect and an income effect[2].

Remark. **For the interested student only.** If S is a surface[3] embedded in $\mathbb{R}^N$, we can apply the IFT locally under mild assumptions. The course Real Analysis 1 studies this.

[2] This result gains interpretive value when deriving the Slutsky equation.

[3] An m-dimensional C^r ($r \geq 1$) parameterization of a set $U \subset \mathbb{R}^M$ is a homeomorphism $\varphi : U_0 \to U$ where U_0 is an open set in $\mathbb{R}^m$ and $\varphi : U_0 \to \mathbb{R}^M$ is a **immersion** ($\forall\ x \in U$, df_x is one-to-one) of class C^r.

An m-dimensional C^r ($r \geq 1$) surface in $\mathbb{R}^M$ is a set $M \subset \mathbb{R}^M$ such that in a neighborhood of each point $p \in M$, we can obtain an open set $U \subset M$ and an m-dimensional parameterization $\varphi : U_0 \to U$ of class C^r, $U_0 \subset \mathbb{R}^m$.

Mandatory exercises

Choice under uncertainty

Exercise 1. Manuel has an initial wealth of $w > 0$ (in soles). There exists a risky asset that provides a return of $z \geq 0$ for each sol invested. In scenario ω_H, which occurs with probability $0 < p_H < 1$, the return is $z_H > 1$. In scenario ω_L, which occurs with probability $p_L = 1 - p_H$, the return is $z_L < 1$. Manuel's Bernoulli utility is given by $v(\cdot)$, a differentiable and strictly concave function. Manuel decides how much of his wealth (α) to invest in this asset.

a) Argue why Manuel's problem can be expressed as

$$\max_{0 \leq \alpha \leq w} \; p_L v(w + \alpha(z_L - 1)) + p_H v(w + \alpha(z_H - 1)).$$

b) Derive the first-order conditions to obtain the optimal investment amount α^*.

c) In which case are these conditions sufficient?

d) Consider the case where the expected net return is non-positive, i.e., $\mathbb{E}[z] - 1 \leq 0$. Show that, in this case, the optimal investment is $\alpha^* = 0$.

e) Consider the case where the expected return is positive, i.e., $\mathbb{E}[z] - 1 > 0$. Show that, in this case, the optimal investment satisfies $\alpha^* > 0$.

Solution:

a) We have only two states of nature, ω_H and ω_L. In state ω_L, the payment that Pancho receives for purchasing α units is αz_L. For this, he had to pay α. Hence, his wealth in this state is: $w + \alpha(z_L - 1)$. A similar reasoning leads to his wealth in state ω_H being: $w + \alpha(z_H - 1)$.Thus, his expected utility is given by:

$$U(\alpha) = p_L v(w + \alpha(z_L - 1)) + p_H v(w + \alpha(z_H - 1)).$$

Finally, we must maximize $U(\alpha)$ over the interval $[0, w]$ since the amount invested must be non-negative and cannot exceed the total wealth (no borrowing is allowed).

b) For interior solutions, we differentiate with respect to α and set it equal to zero:

$$\frac{dU}{d\alpha} = (z_L - 1) p_L v'(w + \alpha(z_L - 1)) + (z_H - 1) p_H v'(w + \alpha(z_H - 1)) = 0.$$

c) The sufficiency of the conditions depends on the concavity of the objective function and the interiority of the solutions (in order to differentiate). Assuming an interior solution, if the second derivative of the objective function is non-positive, we can conclude that the critical point is optimal. Thus, we differentiate U twice:

$$\frac{d^2U}{d\alpha^2} = \underbrace{(z_L-1)^2 p_L}_{\geq 0} \underbrace{v''(w+\alpha(z_L-1))}_{\leq 0} + \underbrace{(z_H-1)^2 p_H}_{\geq 0} \underbrace{v''(w+\alpha(z_H-1))}_{\leq 0} \leq 0.$$

Since $v''(\cdot) \leq 0$, we have $\frac{d^2U}{d\alpha^2} \leq 0$. Thus, the first-order conditions are sufficient under the given assumptions. In the case of a corner solution, i.e., $\alpha^* \in \{0, w\}$, we must compare the objective function evaluated at these points with the objective function evaluated at the critical points. Moreover, if Inada conditions are imposed on $u(\cdot)$, we can rule out $\alpha^* = 0$.

d) To prove that the optimal investment is $\alpha^* = 0$, it is enough to show that $U'(0) \leq 0$. Indeed, if $U'(0) \leq 0$, since $U'(\cdot)$ is monotonic, the maximum is achieved at $\alpha^* = 0$ (given the constraint $\alpha \in [0, w]$). Thus,

$$\begin{aligned}
\frac{dU}{d\alpha}(0) &= (z_L-1)p_L v'(w) + (z_H-1)p_H v'(w) \\
&= (p_L z_L + p_H z_H - p_L - p_H)v'(w) \\
&= \underbrace{(E[z]-1)}_{\leq 0} \underbrace{v'(w)}_{>0} \\
&\leq 0.
\end{aligned}$$

e) By a reasoning similar to the one made in part (d), it is enough to show that $U'(0) > 0$. Indeed, if this is the case, the optimum is found to the right of $\alpha = 0$. Thus,

$$\begin{aligned}
\frac{dU}{d\alpha}(0) &= (z_L-1)p_L v'(w) + (z_H-1)p_H v'(w) \\
&= (p_L z_L + p_H z_H - p_L - p_H)v'(w) \\
&= \underbrace{(E[z]-1)}_{>0} \underbrace{v'(w)}_{>0} \\
&> 0.
\end{aligned}$$

Exercise 2. An individual working in the construction sector receives a wage of 500 soles. However, they are exposed to falls with a probability of $\frac{1}{2}$, which could cost them 100 soles for recovery. Therefore, they wish to insure themselves with a company for an amount M in case of a fall. The individual's utility function is $v(x) = x^{1/2}$. The insurance covers the full cost of 100 soles.

- Find the expected value and the expected utility of the individual without insurance.
- Calculate the maximum amount that a monopolistic insurer can charge for the insurance (find the amount M).

Solution:

- We build the table with payments and probabilities:

	Suffers fall	**Doesn't suffer fall**
Without insurance	400	500
With insurance	$500 - M$	$500 - M$
Probabilities	$\frac{1}{2}$	$\frac{1}{2}$

Expected value

$$\mathbb{E} = \frac{1}{2} \times 400 + \frac{1}{2} \times 500 = 450$$

Expected utility without insurance

$$U^e_{\text{without insurance}} = \frac{1}{2} \times (400)^{1/2} + \frac{1}{2} \times (500)^{1/2} = 21.180$$

- Equating the expected utility without insurance to the expected utility with insurance:

$$21.180 = \sqrt{500 - M}$$

Squaring both sides:

$$(21.180)^2 = 500 - M$$

$$M = 500 - (21.180)^2$$

$$M \approx 51.4076$$

Risk aversion

Exercise 3. Regarding Absolute Risk Aversion (ARA) and Relative Risk Aversion (RRA):

(a) Explain the difference between Absolute Risk Aversion (ARA) and Relative Risk Aversion (RRA). How are they defined mathematically, and what is their economic interpretation?

(b) Consider two individuals with utility functions $v_1(c)$ and $v_2(c)$. The first exhibits constant absolute risk aversion, while the second exhibits constant relative risk aversion. Derive the expressions for $A(c)$ and $R(c)$ for each one.

(c) Suppose an investor has a utility function $v(c;\theta) = \frac{c^{1-\theta}-1}{1-\theta}$, where $\theta \in (0,1)$ is the coefficient of relative risk aversion. Derive the expressions for ARA and RRA for this utility function and explain how they are related. What happens to these measures as wealth c increases?

(d) Calculate $\lim_{\theta \to 1} v(c;\theta)$.

Solution:

a) Remember that,

$$ARA(x) = -\frac{v''(x)}{v'(x)}$$

$$RRA(x) = -x\frac{v''(x)}{v'(x)}$$

Where v is the utility, v' is the marginal utility and v'' measures how concave the utility function is.

Risk aversion is closely linked to the concavity of the elementary utility function v. In the presence of risk aversion (concavity), a loss of $\epsilon > 0$ is not compensated by a gain of ϵ.

However, we do not use $-v''(x)$ directly, because this measure is not invariant under affine linear transformations $av(x) + b$. For instance, while $10\sqrt{x}$ and $\sqrt{x}$ have the same essence regarding risk aversion, the former would have a higher $-v''$ coefficient. Therefore, the Arrow-Pratt coefficient $ARA(x)$ is introduced. Regarding $RRA(x)$, the factor x serves to adjust for the level of wealth.

b) Starting with the first consumer, constant absolute risk aversion means that

$$A(c) = -\frac{v''(c)}{v'(c)} = \gamma$$

Thus,

$$R(c) = -c\frac{v''(c)}{v'(c)} = \gamma c$$

Furthermore, we can compute for $v_1(c)$ by solving the first expression as an ordinary differential equation. If $f(c) = v'(c)$, then $f'(c) = v''(c)$, so you have a separable ordinary differential equation:

$$\frac{f'(c)}{f(c)} = -\gamma$$

$$\frac{df(c)}{f(c)} = -\gamma dc$$

$$\int \frac{df(c)}{f(c)} = \int -\gamma dc$$

$$\ln f = -\gamma c + B$$
$$v'(c) = f(c) = Ae^{-\gamma c}$$
$$v(c) = \int Ae^{-\gamma c} dc$$
$$v_1(c) = Ae^{-\gamma c} + B$$

With A, B, γ constant.

In the case of constant relative risk aversion, we have

$$R(c) = -c\frac{v''(c)}{v'(c)} = \rho$$

Thus,

$$A(c) = -\frac{v''(c)}{v'(c)} = \frac{\rho}{c}$$

Then we can solve,

$$v'(c) = \exp\left[\int -\frac{\rho}{c} dc\right] = \exp(-\rho \ln c + B)$$

So, $v(c) = \int v'(c)dc$, thus,

$$v_2(c) = \frac{Ac^{1-\rho}}{1-\rho} + B, \ \rho \neq 1$$

If $\rho = 1$, then $v_2(c) = A \ln c + B$

c) Directly calculating, we get

$$ARA(c) = \frac{\theta}{c}$$
$$RRA(c) = \theta$$

d) Finally,

$$\lim_{\theta \to 1} = \frac{c^{1-\theta} - 1}{1-\theta} = \lim_{\theta \to 1} \frac{e^{(1-\theta)\ln c} - 1}{1-\theta}$$

Applying l'Hôpital's rule,

$$\lim_{\theta \to 1} \frac{e^{(1-\theta)\ln c} - 1}{1-\theta} = \ln c$$

Comparative statics

Exercise 4. *Based on Allingham and Sandmo (1972).* In the Allingham and Sandmo (1972) model, an individual has wealth w, known only to them. Since the tax authority does not know this information, the individual has the option to declare all their income or an amount smaller than their true income. When the individual reports their income, the state collects a fraction θ of the reported wealth. However, if the individual declares less than their true wealth, there is a probability π that tax authorities will investigate and discover the discrepancy. In such a case, the individual is penalized with a higher tax rate γ, greater than θ, on the undeclared amount. Conversely, with probability $1-\pi$, no investigation occurs, and the individual keeps part of their wealth untaxed. To determine the amount of wealth reported (x), the individual solves the following problem: maximize their expected utility U_e by choosing x such that:

$$\max\ (1-\pi)\cdot v(w-\theta x)+\pi\cdot v(w-\theta x-\gamma(w-x))$$

where v is the Bernoulli utility function. Taking the derivative with respect to x and setting it to zero, we obtain:

$$-(1-\pi)\,\theta v'(w-\theta x)+\pi\,(\gamma-\theta)\,v'(w-\theta x-\gamma(w-x))=0.$$

If the solution is interior, the undeclared amount at equilibrium will be neither zero nor maximum (equal to wealth). Thus, the first-order condition evaluated at income levels close to 0 must be positive, while the condition evaluated at w must be negative.

To illustrate some effects on the amount declared with these variables, we can assume a natural logarithmic utility function. The expected utility maximization problem then takes the form:

$$\max\ (1-\pi)\ln(w-\theta x)+\pi\ln(w-\theta x-\gamma(w-x)).$$

Solve this problem. Additionally, perform comparative statics of x with respect to θ and w, without assuming $v=\ln$, to examine the general case.

Solution:

With a probability of π, they get $w-\theta x-\gamma(w-x)$ and with a probability of $1-\pi$, they get $w-\theta x$. Therefore,

$$U^e=\pi v(w-\theta x-\gamma(w-x))+(1-\pi)v(w-\theta x)$$

If we maximize that, FOC yields

$$\frac{dU^e}{dx}=(\gamma-\theta)\pi v'(w-\theta x-\gamma(w-x))-\theta(1-\pi)v'(w-\theta x)=0$$

Thus,

$$\frac{(\gamma-\theta)\pi}{\theta(1-\pi)}=\frac{v'(w-\theta x)}{v'(w-\theta x-\gamma(w-x))}$$

We verify that it is a maximum, we derive second order conditions,

$$\frac{d^2U^e}{dx^2}=\underbrace{(\gamma-\theta)^2\pi}_{\geq 0}\underbrace{v''(w-\theta x-\gamma(w-x))}_{\leq 0}+\underbrace{\theta^2(1-\pi)}_{\geq 0}\underbrace{v''(w-\theta x)}_{\leq 0}\leq 0$$

We also analyse what happens in the edges,

$$\lim_{x\to 0^+} \frac{dU^e}{dx} = (\gamma-\theta)\pi v'(w-\gamma w) - \theta(1-\pi)v'(w) > 0$$

and

$$\lim_{x\to w^-} \frac{dU^e}{dx} = (\gamma-\theta)\pi v'(w-\theta w) - \theta(1-\pi)v'(w-\theta w) < 0$$

.

If $v(x) = \ln(x)$, then FOC yields

$$\frac{(\gamma-\theta)\pi}{\theta(1-\pi)} = \frac{v'(w-\theta x)}{v'(w-\theta x-\gamma(w-x))}$$

$$\frac{(\gamma-\theta)\pi}{\theta(1-\pi)} = \frac{\frac{-\theta}{w-\theta x}}{\frac{\gamma-\theta}{w-\theta x-\gamma w+\gamma x}}$$

$$-\frac{(\gamma-\theta)\pi}{\theta(1-\pi)} = \frac{\theta(w-\theta x-\gamma w+\gamma x)}{(\gamma-\theta)(w-\theta x)}$$

$$-\frac{\pi}{(1-\pi)} = \frac{\theta^2(w-\theta x-\gamma w+\gamma x)}{(\gamma-\theta)^2(w-\theta x)}$$

Thus,

$$x^* = -\left[\frac{\theta^2-\theta^2\gamma+\pi\theta^2\gamma-2\pi\theta\gamma+\pi\gamma^2}{\theta\left(\theta-\pi\gamma\right)\left(\theta-\gamma\right)}\right] w$$

So,

$$\frac{\partial x}{\partial w} = -\left[\frac{\theta^2-\theta^2\gamma+\pi\theta^2\gamma-2\pi\theta\gamma+\pi\gamma^2}{\theta\left(\theta-\pi\gamma\right)\left(\theta-\gamma\right)}\right]$$

In general, to analyse the effect of w on x, let $y = w-\theta x$ and $z = w-\theta x-\gamma(w-x)$

$$\begin{aligned}
\frac{\partial x}{\partial w} &= \frac{\theta(1-\pi)v''(y)+(\theta-\gamma)(1-\gamma)\pi v''(z)}{\underbrace{\theta^2(1-\pi)v''(y)+(\theta-\gamma)^2\pi v''(z)}_{D}} \\
&= -\frac{\theta(1-\pi)v'(y)}{D}\left[-\frac{v''(y)}{v''(z)}+(1-\gamma)\frac{v''(z)}{v'(z)}\right] \\
&= -\frac{\theta(1-\pi)v'(y)}{D}\left[R_A(y)-(1-\gamma)R_A(z)\right].
\end{aligned}$$

Observation

We know that $RA(y) \leq RA(z)$. However, unless $\gamma \geq 1$, we cannot determine the sign of $\frac{\partial x}{\partial w}$. If $\gamma \geq 1$, then $\frac{\partial x}{\partial w} \geq 0$.

$$\frac{\partial(x/w)}{\partial w} = -\frac{\theta(1-\pi)v'(y)}{w^2 D}\underbrace{\left[R(y)-R(z)\right]}_{\Delta R}.$$

Thus, the variation in the declared fraction (whether it increases, remains constant, or decreases) depends solely on ΔR.

Suggested exercises

Exercise 5. Manuel has a Bernoulli utility function given by $v_M(x) = \sqrt{x}$, while Carlos's utility function is $v_C(x) = \ln x$. Both functions are defined for all $x > 0$.

1. Is Manuel more risk-averse than Carlos? Justify.

2. Consider the following situation. There are two states of the world: the bad state occurs with probability $1/2$, and the good state occurs with the complementary probability. Both Manuel and Carlos have an initial wealth of $w > 0$ soles. Wealth remains at its original level in the good state. However, if the bad state occurs, both suffer a loss of $\ell = w$ (i.e., the loss in the bad state is total). Before the state of nature is known, Manuel and Carlos decide how many units of insurance to buy. One unit of insurance costs t soles, where $1/2 < t < 1$, and pays one sol if the bad state occurs. Solve for how many units Manuel and Carlos each purchase. Compare and conclude who would be preferred as a client in a non-competitive world where clients can be selected. **Note:** if insurance is purchased, the premium is paid in any state.

Exercise 6. Consider a decision-maker with an initial wealth of w who may lose 1 unit of wealth with probability p. This individual can purchase insurance, which is a divisible good. One unit of insurance costs q and covers one unit of loss if it occurs. We wish to understand their demand for insurance. Let θ denote the amount of insurance they purchase.

1. Argue why their expected utility is given by

$$v(w - q\theta)(1-p) + v(w - q\theta - (1-\theta))p,$$

where $v(\cdot)$ is the basic utility function.

2. Consider the case of an actuarially unfair price, where $q > p$. This scenario is common because the insurance company needs to cover its operating costs. Under these conditions, show that the decision-maker buys only partial insurance, that is, $\theta < 1$. Assume that v is strictly increasing and strictly concave (reasonable assumptions).

3. Now consider the case of $q = p$, the actuarially fair price. This price is significant in the literature as it represents the competitive price, assuming insurance companies have no additional costs. In this scenario, show that the decision-maker buys full insurance ($\theta = 1$).

Exercise 7. **The continuous case of Exercise 1.** Consider an investor with an initial wealth w. There is a risky asset that provides a return of z per dollar invested. Let F be the cumulative distribution function (CDF) of z. Let α denote the amount invested in the risky asset. Let $v(\cdot)$ represent the investor's basic utility function. You are asked to:

1. Determine the investor's expected utility.

$$\int v(w + \alpha(z-1))\, dF(z).$$

2. Derive the first-order condition for the optimal level of investment α^*.

3. Consider the case where the expected net return is non-positive, i.e., $\mathbb{E}[z] - 1 \leq 0$. Show that, in this case, the optimal investment is $\alpha^* = 0$.

4. Consider the case where the expected net return is positive, i.e., $\mathbb{E}[z] - 1 > 0$. Show that, in this case, the investment $\alpha = 0$ is not optimal.

5. Demonstrate that if the investor has a higher degree of risk aversion, they will invest less in the risky asset. To do this, consider two investors with Bernoulli utility functions v_1 and v_2, where $v_1 = g \circ v_2$ with g concave and increasing. Show that the optimal investment level α_1^* for v_1 is less than or equal to the optimal investment level α_2^* for v_2.

Hint: recall that, under mild assumptions,[4]

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)dt. \quad (1)$$

Equation (1) is known as Leibniz rule.

Exercise 8. *Adapted from MIT Microeconomic Theory III.* Ann has constant absolute risk aversion $\gamma > 0$ and an initial wealth w. She can buy shares of two divisible assets sold at unit price. One of the assets pays a dividend $X \sim N(2\mu, \sigma^2)$ and the other pays a dividend $Y \sim N(\mu, \sigma^2)$, where X and Y are independently distributed and $\mu > 1$. She can buy any amount of shares of each asset and may hold part of her initial wealth in cash. Find the optimal portfolio for Ann.

Exercise 9. *Adapted from Mas-Colell, Whinston and Green 1995.* Consider a risk preference $\succeq$ that has an expected utility representation with a continuous, increasing Bernoulli utility function u. Prove that the following statements are equivalent:

1. $\succeq$ exhibits risk aversion.

2. $CE_{\succeq}(X) \leq \mathbb{E}[X]$ for any random variable X, where $CE_{\succeq}(X)$ is the certainty equivalent, i.e., the sure value such that the agent is indifferent between receiving this value or facing the lottery X.

3. $RP_{\succeq}(X) \geq 0$ for any random variable X, where $RP_{\succeq}(X)$ is the risk premium, i.e., the maximum amount an individual would be willing to pay to eliminate the risk associated with the lottery X.

4. $\int u(x)\, dF(x) \leq u\left(\int x\, dF(x)\right)$ for any distribution F.

[4] $f(x,t)$, $f_x(x,t)$ continuous, and $a(x), b(x)$ C^1.

5. The function u is concave.

Exercise 10. *Adapted from Mas-Colell, Whinston and Green 1995.* Consider two individuals choosing between two monetary lotteries. Define that the utility function $v^*(\cdot)$ is strongly more risk-averse than $v(\cdot)$ if and only if there exists a positive constant k and a **non-increasing** concave function $g(\cdot)$ such that $v^*(x) = kv(x) + g(x)$ for all x. Show that if $v^*(\cdot)$ is strongly more risk-averse than $v(\cdot)$, then $v^*(\cdot)$ is more risk-averse than $v(\cdot)$ in the usual Arrow-Pratt sense. **Hint:** compare the Arrow-Pratt coefficients of both utility functions.

Exercise 11. *Adapted from Varian 1992.* A consumer faces two risks and can only eliminate one of them. That is, let $\tilde{w} = w_1$ with probability $p > 0$ and $\tilde{w} = w_2$ with probability $1 - p$. Let $\tilde{\epsilon} = 0$ if $\tilde{w} = w_2$, and, if $\tilde{w} = w_1$, then $\tilde{\epsilon} = \epsilon$ with probability $1/2$ and $\tilde{\epsilon} = -\epsilon$ with probability $1/2$. Define the risk premium π_v associated with $\tilde{\epsilon}$ as the number such that

$$\mathbb{E}[v(\tilde{w} - \pi_v)] = \mathbb{E}[v(\tilde{w} + \tilde{\epsilon})].$$

1. Show that if $\epsilon \to 0$,
$$\pi_v = \frac{-\frac{1}{2}pv''(w_1)\epsilon^2}{pv'(w_1) + (1-p)v'(w_2)}. \tag{2}$$
Hint: consider a first-order Taylor expansion for the term on the left side of (2) and a second-order expansion for the term on the right side.

2. Let $v_1(w) = e^{-aw}$ and $v_2(w) = e^{-bw}$. Calculate the Arrow-Pratt coefficient (ARA) of v_1 and v_2.

3. Suppose that $a > b$. Show that if $p < 1$, there exists a sufficiently large value of $w_1 - w_2$ such that $\pi_{v_2} > \pi_{v_1}$.

Exercise 12. *Adapted from Gallardo 2018.* A question of great interest is whether the expected utility U^e depends, in certain cases, solely on $\mathbb{E}[X] = \mu$ and $\mathbb{E}[X^2] = \sigma^2$ (regardless of other parameters of the model). Here, X is a random variable. Recall that

$$U^e = \begin{cases} \displaystyle\sum_{i=1}^{N} p_i u(x_i) & \text{(discrete distribution case)} \\ \displaystyle\int_a^b u(x) f(x)\, dx & \text{(continuous distribution case)} \end{cases}$$

Below, two situations are presented in which U^e depends solely on μ and σ^2. One corresponds to a discrete distribution, and the other to a continuous distribution. Consider

$$u(x) = k_0 + k_1 x - \frac{k_2}{2}x^2,\ k_1, k_2 > 0,\ k_0 \in \mathbb{R}. \tag{3}$$

1. Show how the quadratic utility function given by (3) results in an expression for U^e that depends solely on the mean μ and variance σ^2 of the discrete distribution.

2. Explain the relationship between the derivative of the mean $\frac{d\mu}{d\sigma}$ and risk aversion. **Hint:** prove that

$$\frac{d\mu}{d\sigma} = \frac{\sigma k_2}{k_1 - k_2\mu}, \; k_1 - k_2\mu \neq 0.$$

3. For the continuous case, show that

$$\begin{aligned} U^e &= -\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu+a\sigma^2)^2}{2\sigma^2}} e^{\frac{-2\mu a\sigma^2 + a^2\sigma^4}{2\sigma^2}} \, dx \\ &= -e^{\frac{a^2\sigma^2 - 2\mu a}{2}}. \end{aligned}$$

when the utility function is $u(x) = -e^{-ax}$ and the distribution $f(x)$ is normal with mean μ and variance σ^2.

Solution:

$$U_e = \sum_{i=1}^{N} p_i u(x_i)$$

$$= \sum_{i=1}^{N} p_i \left(u(\mu) + u'(\mu)(x_i - \mu) + \frac{u''(\mu)}{2!}(x_i - \mu)^2 \right)$$

$$= u(\mu) + \frac{u''(\mu)}{2}\sigma^2 = k_0 + k_1\mu - \frac{k_2}{2}(\mu^2 + \sigma^2).$$

We also note that:

$$\frac{d\mu}{d\sigma} = \frac{\sigma k_2}{k_1 - k_2\mu},$$

which implies that increases in σ generate more than proportional increases in μ. This is a consequence of risk aversion. Moving to the continuous context:

$$U_e = \int_{\mathbb{R}} u(x) f(x) \, dx, \quad u(x) = -e^{-ax}, \quad a > 0, \quad f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Thus:

$$\begin{aligned} U_e &= \int_{\mathbb{R}} -e^{-ax} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \\ &= -\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-2\sigma^2 ax + x^2 - 2x\mu + \mu^2}{2\sigma^2}} \, dx. \end{aligned}$$

Rearranging:

$$x^2 + \mu^2 + 2(\sigma^2 a - \mu) = x^2 - 2(\mu - \sigma^2 a)x + \mu^2 = (x - \mu + a\sigma^2)^2 + 2\mu a\sigma^2 - a^2\sigma^4,$$

we conclude that:

$$
\begin{aligned}
U_e &= -\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu+a\sigma^2)^2}{2\sigma^2}} e^{\frac{-2\mu a\sigma^2+a^2\sigma^4}{2\sigma^2}}\, dx \\
&= -e^{\frac{a^2\sigma^2-2\mu a}{2}}.
\end{aligned}
$$

Advanced exercises

Exercise 13. *Adapted from Mas-Colell, Whinston and Green 1995.* Consider a decision-maker with a utility function $u(\cdot)$ defined over $\mathbb{R}^L_+$.

a) Argue why the concavity of $u(\cdot)$ can be interpreted as the decision-maker exhibiting risk aversion with respect to lotteries whose outcomes are bundles of the L basic goods (commodities).

b) Now suppose that a Bernoulli utility function $v(\cdot)$ for wealth is derived from the maximization of a utility function defined over commodities for each given level of wealth w, with commodity prices fixed. Show that if the utility function for commodities exhibits risk aversion, then the derived Bernoulli utility function for wealth also exhibits risk aversion. Interpret.

c) Argue why the converse of part (b) need not be true. That is, there are non-concave functions $u : \mathbb{R}^L_+ \to \mathbb{R}$ such that, for any price vector, the Bernoulli utility function for wealth exhibits risk aversion.

Exercise 14. *Adapted from Mas-Colell, Whinston and Green 1995.* Suppose we have N risky assets whose returns z_n ($n = 1, \ldots, N$) per dollar invested are jointly distributed according to the distribution function $F(z_1, \ldots, z_N)$. Assume also that all returns are non-negative with probability one. Consider an individual with a continuous, increasing, and concave Bernoulli utility function $v(\cdot)$ over $\mathbb{R}_+$. Define the expected utility function $U^e(\cdot)$ of this investor over $\mathbb{R}^N_+$, the set of all non-negative portfolios, as follows:

$$U^e(\alpha_1, \ldots, \alpha_N) = \int v(\alpha_1 z_1 + \cdots + \alpha_N z_N)\, dF(z_1, \ldots, z_N).$$

Show that $U^e(\cdot)$ is:

- Increasing.
- Concave.
- Continuous.

Hint: apply dominated convergence theorem for the last item.

Exercise 15. *Adapted from IMPA, Introduction to Mathematical Economics lecture notes.* Consider an investor who is constructing a portfolio composed of k assets. The variance of the portfolio return is given by Var $\left(\sum_{i=1}^k \alpha_i A_i\right)$, where α_i is the proportion invested in asset i, A_i is the return of asset i, and Σ is the covariance matrix of the asset returns. Formulate and solve the portfolio risk minimization problem. Remember that the expected return is equal to μ_0 and that the sum of the invested proportions is equal to 1. Use the method of Lagrange multipliers to find the optimal proportions α_i.