FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Recitation 5

Microeconomics 2
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1 Monopoly

Exercise 1.1. An economy has two types of consumers and two goods. The agent type A has the following utility function:

$$u_A(x_{1A}, x_{2A}) = 4x_{1A} - \frac{x_{1A}^2}{2} + x_{2A}$$

and the agent type B has the following utility function:

$$u_B(x_{1B}, x_{2B}) = 3x_{1B} - \frac{x_{1B}^2}{2} + x_{2B}.$$

Good 2 is the numeraire, and each consumer has an income of 100. Additionally, the economy has N consumers of both type A and type B.

1. Identify the type of consumer with high demand and the type with low demand for good x_1. Compare the marginal willingness to pay for each type of consumer for good x_1.

   Solution:
   First, we can break down the utility functions in the following way so that we can compare them:

$$u_A(x_{1A}, x_{2A}) = \underbrace{4x_{1A} - \frac{x_{1A}^2}{2}}_{u_A(x_{1A})} + x_{2A}$$

$$u_B(x_{1B}, x_{2B}) = \underbrace{3x_{1B} - \frac{x_{1B}^2}{2}}_{u_B(x_{1B})} + x_{2B}$$

Thus, we can see that

$$u_A(x_1) > u_B(x_1)$$

And, we can also compare the marginal willingness to pay

$$\underbrace{4-x}_{u'_A(x_1)} > \underbrace{3-x}_{u'_B(x_1)}$$

So, we can conclude that the type of consumer with **high demand** is type A and the type of consumer with **low demand** is type B.

2. The monopolist produces good 1 with the following cost function $C(x_1) = cx_1$ and cannot discriminate prices. Find the optimal price and quantity of good x_1 that the monopolist will choose. For which values of c will the monopolist choose to sell to both types of consumers?

Solution:
First, we find each consumer's demand.

For type A:

$$\begin{aligned} &\max\ u_A(x_{1A}, x_{2A}) = 4x_{1A} - \frac{x_{1A}^2}{2} + x_{2A} \\ &\text{s.t. } px_{1A} + x_{2A} = 100 \end{aligned}$$

The Lagrangian would then be:

$$\mathscr{L} = 4x_{1A} - \frac{x_{1A}^2}{2} + x_{2A} + \lambda(100 - px_{1A} - x_{2A})$$

Applying FOCs,

$$\frac{\partial \mathscr{L}}{\partial x_{1A}} = 4 - x_{1A} - \lambda p = 0 \tag{1}$$
$$\frac{\partial \mathscr{L}}{\partial x_{2A}} = 1 - \lambda = 0 \tag{2}$$

Thus, $\lambda = 1$, so, inputing in (1)

$$x_{1A} = 4 - p$$

For type B, we follow the same steps:

$$\begin{aligned} &\max\ u_B(x_{1B}, x_{2B}) = 3x_{1B} - \frac{x_{1B}^2}{2} + x_{2B} \\ &\text{s.t. } px_{1B} + x_{2B} = 100 \end{aligned}$$

The Lagrangian would then be:

$$\mathscr{L} = 3x_{1B} - \frac{x_{1B}^2}{2} + x_{2B} + \lambda(100 - px_{1B} - x_{2B})$$

Applying FOCs,

$$\frac{\partial \mathscr{L}}{\partial x_{1B}} = 3 - x_{1B} - \lambda p = 0 \tag{3}$$

$$\frac{\partial \mathscr{L}}{\partial x_{2B}} = 1 - \lambda = 0 \tag{4}$$

Thus, $\lambda = 1$, so, inputing in (3)

$$x_{1B} = 3 - p$$

With that, we can add up for the N individuals there are in each sector to find the aggregate demand:

$$X_1 = Nx_{1A} + Nx_{1B} = \sum_{i=1}^{N}(4 - p) + \sum_{i=1}^{N}(3 - p)$$

$$X_1 = N(7 - 2p)$$

Thus, we can solve for the price and find the inverse demand function

$$p(X_1) = \frac{7}{2} - \frac{X_1}{2N}$$

We do this so that we can solve the problem for the monopoly, which is

$$\max \Pi = p(X_1)X_1 - C(X_1)$$

We can substitute the inverse demand function we have and the cost function that was given to us, so we end up with

$$\max \Pi = \left(\frac{7}{2} - \frac{X_1}{2N}\right) X_1 - cX_1$$

Applying FOCs, we get

$$\frac{\partial \Pi}{\partial X_1} = \frac{7}{2} - \frac{X_1}{N} - c = 0$$

So,

$$
\begin{aligned}
X_1^* &= \frac{7}{2} - Nc \\
p^* &= \frac{7}{2} - \frac{X_1^*}{2N} \\
p^* &= \frac{7}{4} + \frac{c}{2}
\end{aligned}
$$

Note that the monopolist cannot incur in price discrimination, which means that he chooses to sell at one price. As we had observed, the maximum price that B (the low demand type) was willing to pay was 3, so for the monopolist to sell to both types of consumers, c would have to be such that $p \leq 3$. That is,

$$
\begin{aligned}
\frac{7}{4} + \frac{c}{2} \leq&\ 3 \\
\frac{c}{2} \leq&\ 1.25 \\
c \leq&\ 2.5
\end{aligned}
$$

3. The monopolist engages in second-degree price discrimination by offering a menu of prices and quantities to each type of consumer (r_A, x_A) and (r_B, x_B). Based on this, formulate the monopolist's optimization problem and find the optimal values (r_A^*, x_A^*) and (r_B^*, x_B^*).

   Solution:
   In second degree price discrimination, the monopolist looks for (r_A^*, x_A^*) and (r_B^*, x_B^*), where r_i is the total amount charged (that corresponds with the maximum willingness to pay). Concretely, the monopolist solves:

$$
\begin{aligned}
\max\ & N(r_A - cx_{1A}) + N(r_B - cx_{1B}) \\
\text{s.t. } & u_A(x_{1A}) \geq r_A \\
& u_B(x_{1B}) = r_B \\
& u_A(x_{1A}) - r_A = u_A(x_{1B}) - r_B \\
& u_B(x_{1B}) - r_B \geq u_B(x_{1A}) - r_A
\end{aligned}
$$

As we have discussed, the type of consumer with low demand, will not want the high demand package, so our focus is to have them participate in the market. On the other hand, the type of consumer with the high demand might pick either package, so we need to ensure that they'll pick the selected one by making them indifferent between the two packages. From the binding constraints, then, we can solve for r_A and r_B and input that in the utility function:

$$
\begin{aligned}
r_B =&\ u_B(x_{1B}) \\
r_A =&\ u_A(x_{1A}) - u_A(x_{1B}) + r_B \\
r_A =&\ u_A(x_{1A}) - u_A(x_{1B}) + u_B(x_{1B})
\end{aligned}
$$

Thus, we maximize

$$N[u_A(x_{1A}) - u_A(x_{1B}) + 2u_B(x_{1B}) - cx_{1A} - cx_{1B}]$$

With the given utilities then

$$\max_{x_{1A},x_{1B}} N\left[4x_{1A} - \frac{x_{1A}^2}{2} - 4x_{1B} - \frac{x_{1B}^2}{2} + 2\left(3x_{1B} - \frac{x_{1B}^2}{2}\right) - cx_{1A} - cx_{1B}\right]$$

Applying FOCs,

$$\begin{aligned}\frac{\partial \Pi}{\partial x_{1A}} &= 4 - x_{1A} - c = 0 \\ \frac{\partial \Pi}{\partial x_{1B}} &= -4 + x_{1B} + 6 - 2x_{1B} - c = 0\end{aligned}$$

Therefore,

$$\begin{aligned}x_{1A}^* &= 4 - c \\ x_{1B}^* &= 2 - c \\ r_A^* &= u_A(x_{1A}^*) - u_A(x_{1B}^*) + u_B(x_{1B}^*) = \frac{(2-c)(c+4)}{2} \\ r_B^* &= u_B(x_{1B}^*) = 6 + c - \frac{c^2}{2}\end{aligned}$$

4. If the monopolist engages in third-degree price discrimination, what will be the prices and quantities set by the monopolist in the markets for A-type and B-type consumers?

Solution:

For third degree price discrimination, the monopolist solves

$$\max_{x_{1A},x_{1B}} \Pi = \underbrace{\left(4 - \frac{x_{1A}}{N}\right)}_{p_A} x_{1A} + \underbrace{\left(3 - \frac{x_{1B}}{N}\right)}_{p_B} x_{1B} - cx_{1A} - cx_{1B}$$

Applying FOCs, we have

$$\begin{aligned}\frac{\partial \Pi}{\partial x_{1A}} &= 4 - \frac{x_{1A}}{2N} - c = 0 \\ \frac{\partial \Pi}{\partial x_{1B}} &= 3 - \frac{x_{1B}}{2N} - c = 0\end{aligned}$$

Thus,

$$x_{1A}^* = \frac{N(4-c)}{2}$$
$$x_{1B}^* = \frac{N(3-c)}{2}$$

Finally, we can substitute in the inverse demands to find the prices

$$p_A^* = 4 - \frac{x_{1A^*}}{N} = \frac{4+c}{2}$$
$$p_B^* = 3 - \frac{x_{1B^*}}{N} = \frac{3+c}{2}$$

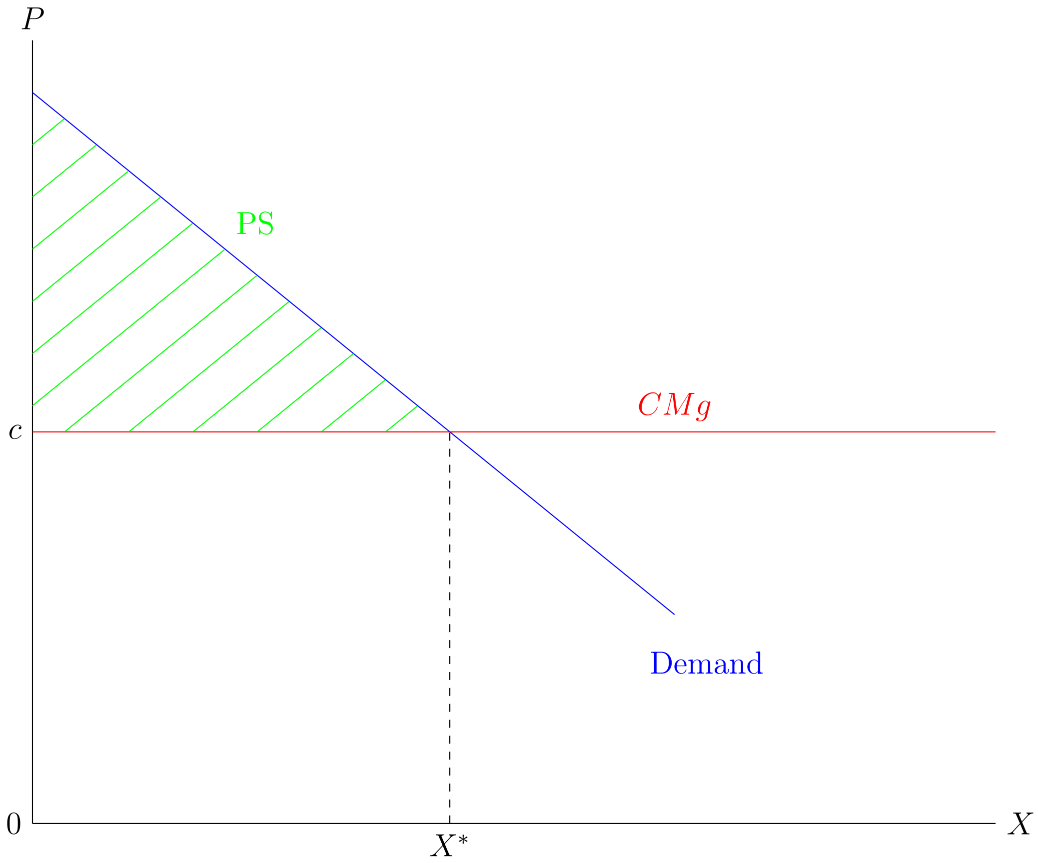
5. If the monopolist engages in first-degree price discrimination, find the quantity produced by the monopolist in the market for good x. Calculate the consumer surplus and the monopolist's surplus.

   Solution:
   In the case of first degree price discrimination, each consumer pays their maximum willingness to pay up till the point of the competitive equilibrium. In this case, that point would be where $p = CMg$, therefore, in this case, $\frac{7}{2} - \frac{X_1}{2N} = c$

$$(X^*, p^*) = \left(N(7-2c),\ \frac{7}{2} - \frac{X}{2N}\right)$$

   Then, consumer surplus would be 0 meanwhile for the producer

$$\begin{aligned} PS &= \int_0^{X^*} p(X_A)dX_A + \int_0^{X^*} p(X_B)dX_B - \int_0^{X^*} CMg\ dX \\ &= \int_0^{X^*} p(X)dX - \int_0^{X^*} c\ dX \\ &= \int_0^{N(7-2c)} \left(\frac{7}{2} - \frac{X}{2N}\right) dX - \int_0^{N(7-2c)} c\ dX \\ &= \frac{N(7-2c)^2}{4} \end{aligned}$$



Exercise 1.2. From Tirole (1994). Consider $q(p) = p^{-\epsilon}$ and assume constant marginal cost. Prove that the social welfare in competitive equilibrium is

$$\mathcal{W}^s = \frac{c^{1-\epsilon}}{\epsilon - 1}.$$

Then, compute the welfare loss. *Hint*: recall that the total surplus is in the competitive case

$$\int_{p=c'}^{\infty} q(s)ds.$$

So, in this case, $\int_{p=c}^{\infty} s^{-\epsilon}ds = \frac{c^{1-\epsilon}}{\epsilon-1}$. For the monopolist case, apply FOC

$$p^m = \frac{c}{1-\frac{1}{\epsilon}}.$$

Thus, you can conclude that

$$\mathcal{W}^s - \mathcal{W}^m = \left(\frac{c^{1-\epsilon}}{\epsilon-1}\right)\left[1 - \left(\frac{2\epsilon-1}{\epsilon-1}\right)\left(\frac{\epsilon}{\epsilon-1}\right)^{-\epsilon}\right].$$

Indeed,

$$\mathcal{W}^s = \frac{c^{1-\epsilon}}{\epsilon-1}$$

$$\Pi^m = p^m q^m - cq^m = \left[\frac{c\epsilon}{\epsilon-1}\right]\left[\frac{c\epsilon}{\epsilon-1}\right]^{-\epsilon} - c\left[\frac{c\epsilon}{\epsilon-1}\right]^{-\epsilon} = c^{1-\epsilon}\underbrace{\left[\frac{\epsilon}{\epsilon-1} - 1\right]}_{\frac{1}{\epsilon-1}}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}.$$

and the new consumer surplus is

$$\int_{\frac{c}{1-1/\epsilon}}^{\infty} s^{-\epsilon} ds = (\epsilon-1)^{\epsilon-2}\epsilon^{1-\epsilon}c^{1-\epsilon}.$$

Adding,

$$\underbrace{\int_{\frac{c}{1-1/\epsilon}}^{\infty} s^{-\epsilon} ds}_{\text{new surplus}} + \frac{c^{1-\epsilon}}{\epsilon-1}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon} = \frac{2\epsilon-1}{\epsilon-1}\left(\frac{\epsilon}{\epsilon-1}\right)^{-\epsilon}.$$

Indeed

$$\begin{aligned}
\int_{\frac{c}{1-1/\epsilon}}^{\infty} s^{-\epsilon} ds + \frac{c^{1-\epsilon}}{\epsilon-1}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon} &= (\epsilon-1)^{\epsilon-2}\epsilon^{1-\epsilon}c^{1-\epsilon} + \frac{c^{1-\epsilon}}{\epsilon-1}\frac{\epsilon^{-\epsilon}}{(\epsilon-1)^{-\epsilon}} \\
&= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\left[\frac{\epsilon}{(\epsilon-1)^2}+\frac{1}{\epsilon-1}\right] \\
&= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\left[\frac{\epsilon(\epsilon-1)+(\epsilon-1)^2}{(\epsilon-1)^3}\right] \\
&= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\left(\frac{2\epsilon-1}{(\epsilon-1)^2}\right) \\
&= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\frac{1}{\epsilon-1}\left(\frac{2\epsilon-1}{\epsilon-1}\right).
\end{aligned}$$

2 Externalities

Exercise 2.1. Consider an economy where there are two goods (y_1, y_2), two consumers, and a firm. Each consumer has an initial endowment of four units of good 1 and nothing of good 2. Good y_1 is not producible, and good y_2 is produced by the firm using good 1 as input (the quantities of good 1 not directly consumed by individuals) based on the following production function:

$$y_2 = f(z) = z_1^{1/2},$$

where y_2 is the quantity of good 2 produced and z_1 is the amount of good 1 used as input. The profits earned by the firm are equally distributed between the two consumers, $\theta_j = 1/2$. Both consumers derive utility from the consumption of the two goods. However, the production of good 2 generates noise and pollution, which negatively affects their well-being. As a result, the utility function of the consumers is given by the following expression:

$$u_i(y_{i1}, y_{i2}, y_2) = y_{i1} + \ln y_{i2} - \frac{1}{2}\ln y_2, \quad i = 1, 2.$$

The superscript refers to the consumer.

a) Calculate the quantities of good 2 produced and consumed by the two individuals in the general equilibrium, assuming the price of good 1 as the numéraire, whether used as a consumption good or as an input $(p_1 = w_1 = 1)$.

Solution:
First, we solve the firm's maximization problem

$$\max\ \Pi = p\sqrt{z_1} - \underbrace{w_1}_{1} z_1$$

FOCs give us

$$\frac{\partial \Pi}{\partial z_1} = \frac{p}{2\sqrt{z_1}} - 1 = 0$$

$$\begin{aligned} z_1^d &= \frac{p^2}{4} \\ y_2^s &= \sqrt{z_1} = \frac{p}{2} \end{aligned}$$

So, we get

$$\Pi = p y_2^s - z_1^d = \frac{p^2}{4}$$

Then, for $i = 1, 2$

$$\begin{aligned} \max_{y_{i1}, y_{i2}}\ & u_i(y_{i1}, y_{i2}, y_2) = y_{i1} + \ln y_{i2} - \frac{1}{2} \ln y_2 \\ \text{s.t.}\ & \underbrace{w_1 y_{i1} + p y_{i2}}_{= y_{i1} + p y_{i2}} = 4w + \underbrace{\theta_j \Pi}_{= p^2/8} \end{aligned}$$

Then, the Lagrangian is

$$\mathscr{L} = y_{i1} + \ln y_{i2} - \frac{1}{2} \ln y_2 + \lambda \left(4 + \frac{p^2}{8} - y_{i1} - p y_{i2} \right)$$

Applying FOCs, we have

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial y_{i1}} &= 1 - \lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial y_{i2}} &= \frac{1}{y_{i2}} - \lambda p = 0 \implies \frac{1}{y_{i2}} = p \end{aligned}$$

To clear the market, we must have

$$\begin{aligned} y_{12} + y_{22} &= \frac{2}{p} \\ &= y_2^s \\ &= \frac{p}{2} \end{aligned}$$

Thus, $p = 2$. Finally,

$$y_{12}^* = y_{22}^* = \frac{1}{2},\ z_1^* = 1,\ y_2^* = 1.$$

b) Calculate the quantities of good 2 produced and consumed by the two individuals in the efficient allocation, and comment on the results in comparison to the previous part.

Solution:

With respect to Pareto Optimal Allocations, we must solve

$$\begin{aligned}
\max U^1 &= y_{11} + \ln y_{12} - \frac{1}{2}\ln y_2 \\
s.t.\ U^2 &= y_{21} + \ln y_{22} - \frac{1}{2}\ln y_2 = \overline{U} \\
&\sqrt{z_1} = y_2 \\
&z_1 + y_{11} + y_{21} = 8 \\
&y_{12} + y_{22} = y_2.
\end{aligned}$$

Replacing the restrictions,

$$\begin{aligned}
\max_{y_{21}, y_{22}, y_2, z_1} \quad & (8 - y_{21} - z_1) + \ln(y_2 - y_{22}) - \frac{1}{2}\ln y_2 \\
s.t.\ U^2 &= y_{21} + \ln y_{22} - \frac{1}{2}\ln y_2 = \overline{U} \\
&\sqrt{z_1} = y_2.
\end{aligned}$$

The Lagrangian is

$$\begin{aligned}
\mathscr{L}(y_{21}, y_{22}, y_2, z_1, \lambda, \mu) = (8 - y_{21} - z_1) + \ln(y_2 - y_{22}) - \frac{1}{2}\ln y_2 \\
+ \lambda(\overline{U} - y_{21} - \ln y_{22} + \frac{1}{2}\ln y_2) + \mu(\sqrt{z_1} - y_2).
\end{aligned}$$

FOC lead to

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial y_{21}} &= -1 - \lambda = 0 \\
\frac{\partial \mathscr{L}}{\partial y_{22}} &= -\frac{1}{y_2 - y_{22}} - \frac{\lambda}{y_{22}} = 0 \\
\frac{\partial \mathscr{L}}{\partial y_2} &= \frac{1}{y_2 - y_{22}} - \frac{1}{2y_2} + \frac{\lambda}{2y_2} + \mu = 0 \\
\frac{\partial \mathscr{L}}{\partial z_1} &= 1 - \frac{\mu}{2\sqrt{z_1}} = 0 \\
\frac{\partial \mathscr{L}}{\partial \lambda} &= \overline{U} - y_{21} - \ln y_{22} + \frac{1}{2}\ln y_2 = 0 \\
\frac{\partial \mathscr{L}}{\partial \mu} &= y_2 - \sqrt{z_1} = 0
\end{aligned}$$

From the first equation, $\lambda = -1$. Then,

$$\frac{1}{y_2 - y_{22}} = \frac{1}{y_{22}} \implies y_2 = 2y_{22}.$$

Next, from the third equation,

$$\frac{1}{y_{22}} - \frac{1}{2y_2} - \frac{1}{2y_2} + \mu = 0 \implies \mu = -\frac{1}{2y_{22}} = -\frac{1}{y_2}.$$

Substituting this into the fourth equation, and using that $y_2 = \sqrt{z_1}$,

$$1 = \frac{1}{2\sqrt{z_1}y_2} = \frac{1}{2z_1} \implies z_1^* = \frac{1}{2},\ y_2^* = \frac{1}{\sqrt{2}}.$$

Finally, since $y_{12} + y_{22} = y_2$,

$$\begin{cases} y_{12}^* = \frac{1}{2\sqrt{2}} \\ y_{22}^* = \frac{1}{2\sqrt{2}} \end{cases}$$

Exercise 2.2. The company S produces a certain amount of steel (s) and a certain amount of pollution (x), which is discharged into a river. The company F is a fish farm located downstream and is negatively affected by the pollution from company S. Suppose that the cost function of S involves both s and x. Meanwhile, the company F depends on f, representing the collection of fish, and x, which represents the production of pollution. Additionally, it must be considered that pollution increases the cost of fish production and reduces the cost of steel production.

a) Formulate the profit maximization problem for both companies.

Solution:
The steel firm's maximization problem is

$$\max_{x,s} \ \Pi_S = p_S \cdot s - C_S(x, s),$$

while, for the fishing firm, the problem is

$$\max_{f} \ \Pi_F = p_F \cdot f - C_F(x, f).$$

b) What are the conditions that characterize profit maximization? Remember that polluting has no price.

Solution:
Applying first-order conditions, we obtain

$$\begin{aligned} \frac{\partial \Pi_S}{\partial s} &= p_S - \frac{\partial C_S}{\partial s} = 0 \\ \frac{\partial \Pi_S}{\partial x} &= -\frac{\partial C_S}{\partial x} = 0 \\ \frac{\partial \Pi_F}{\partial f} &= p_F - \frac{\partial C_F}{\partial f} = 0. \end{aligned}$$

This indicates that the steel firm will pollute until the marginal cost of polluting (which is negative because they *save* by polluting) is zero. On the other hand, both the steel and fishing firms will equalize the marginal cost of production, whether for steel or fish, with the market price. Firm S does not take into account the social cost of its pollution. Therefore, firm S is expected to produce too much pollution from a social perspective. In this case, the question arises: **What would the efficient (Pareto) production levels be?** Let's see.

c) How would the efficient production plan of steel and fish in the Pareto sense look? What are the implications of this new scenario for pollution production?

Solution:
Recall that Pareto efficiency implies that: *No one's welfare can be improved without worsening someone else's situation*. To account for this welfare concept, we assume that a single owner possesses both firms, and that the two are merged into one. In other words, to account for this welfare idea, both firms are integrated into a single entity:

$$\max_{x,f,s} \ \Pi = p_s s + p_f f - C_S(s,x) - C_F(f,x).$$

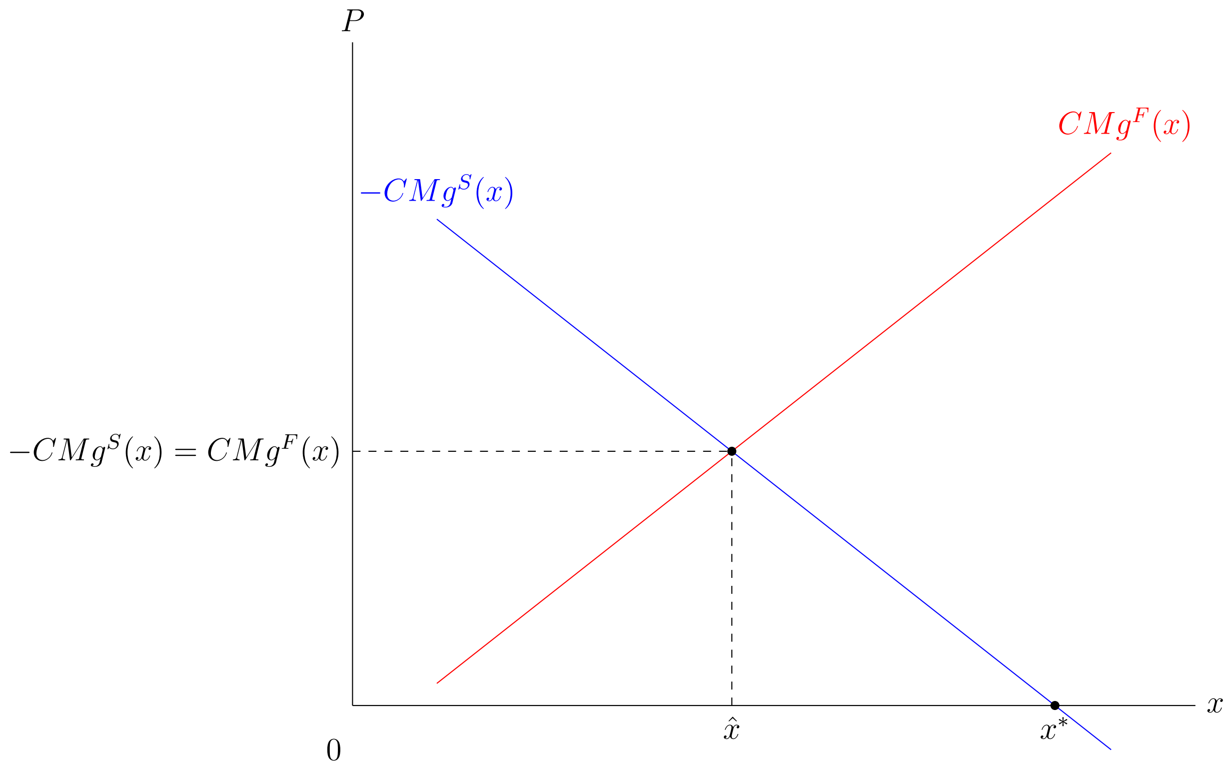
The first-order conditions are

$$\frac{\partial \Pi}{\partial s} = p_s - \frac{\partial C_s}{\partial s} = 0$$
$$\frac{\partial \Pi}{\partial f} = p_f - \frac{\partial C_f}{\partial f} = 0$$
$$\frac{\partial \Pi}{\partial x} = -\frac{\partial C_s}{\partial x} - \frac{\partial C_f}{\partial x} = 0$$

(a) Compared to the individual situations, the only difference is the last equation.

(b) The relevant condition is

$$-\frac{\partial C_s}{\partial x} = \frac{\partial C_f}{\partial x}.$$



Lima, October 7, 2024.

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