FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Recitation 4

Microeconomics 2
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1 Monopoly

We closely follow Tirole (1994). Solutions can be found in such book.

Exercise 1.1. Consider the classical problem of the monopolist. Prove that, if $q = D(p)$ is the demand for the good produced by the monopolist, then the optimal pricing for the monopolist satisfies

$$p^m - C'(D(p^m)) = -\frac{D(p^m)}{D'(p^m)}$$

where $C = C(q)$ is the cost function of the monopolist.

Exercise 1.2. Let ε be the demand elasticity at p^m. Then, prove that

$$\underbrace{\frac{p^m - C'}{p^m}}_{\text{relative markup}} = \underbrace{\frac{1}{\epsilon}}_{\text{Lerner Index}}.$$

Exercise 1.3. Prove that if $q(p) = kp^{-\epsilon}$ with $k, \epsilon > 0$, then Lerner index is constant.

Exercise 1.4. Prove that the monopoly price is a non-decreasing function of the marginal cost $C'(\cdot)$.

Hint: assume that $C_2'(q) > C_1'(q)$ for every q. Denote (p_1^m, q_1^m) and (p_2^m, q_2^m) the optimal pricing-quantities of a monopolist with marginal cost C_1 and C_2 respectively. Then, note that

$$p_1^m q_1^m - C_1(q_1^m) \geq p_2^m q_2^m - C_1(q_2^m) \tag{1}$$

and

$$p_2^m q_2^m - C_2(q_2^m) \geq p_1^m q_1^m - C_2(q_1^m). \tag{2}$$

Add (1) and (2). Then, applying the fundamental theorem of calculus, conclude that

$$\int_{q_2^m}^{q_1^m} (C_2'(q) - C_1'(q))dq.$$

Since $C_2' > C_1'$, $q_1^m > q_2^m$ and therefore $p_2^m > p_1^m$.

Exercise 1.5. Compare the monopolist surplus with the competitive pricing surplus.[1]

Exercise 1.6. Consider $q(p) = p^{-\epsilon}$ and assume constant marginal cost. Prove that the social welfare in competitive equilibrium is

$$\mathcal{W}^s = \frac{c^{1-\epsilon}}{\epsilon - 1}.$$

Then, compute the welfare loss. *Hint*: recall that the total surplus is in the competitive case

$$\int_{p=c'}^{\infty} q(s)ds.$$

So, in this case, $\int_{p=c}^{\infty} s^{-\epsilon}ds = \frac{c^{1-\epsilon}}{\epsilon-1}$. For the monopolist case, apply FOC

$$p^m = \frac{c}{1 - \frac{1}{\epsilon}}.$$

Thus, you can conclude that

$$\mathcal{W}^s - \mathcal{W}^m = \left(\frac{c^{1-\epsilon}}{\epsilon - 1}\right)\left[1 - \left(\frac{2\epsilon - 1}{\epsilon - 1}\right)\left(\frac{\epsilon}{\epsilon - 1}\right)^{-\epsilon}\right].$$

Exercise 1.7. Consider one possible policy (taxation) in order to restore the social optimum in presence of a monopoly. For this, assume that the government taxes monopoly output at rate $t > 0$.

1. Justify that the maximization problem of the monopolist

$$\max_p \{pD(p+t) - C(D(p+t))\}.$$

2. Prove that $t = \frac{D(p^c)}{D'(p^c)} < 0$.

3. Interpret that $t < 0$.

Exercise 1.8. Prove that if the demand function $D = D(p)$ is concave, then the marginal revenue is decreasing. This is, setting $R(p) = pD(p)$, $R''(p) < 0$.

Exercise 1.9. Analyze the following statements:

1. Monopolies usually have large fixed costs.

2. Monopolist do not exhibit (often) cost subadditivity.

[1]Monopolist profits is $p^m q^m - C(q^m)$. Consumer surplus is $\int_0^{q^m} p(q)dq - p^m q^m$. On the other hand,

3. The monopolist operates in the inelastic part of the demand, $|\varepsilon| < 1$.

Exercise 1.10. Consider a quasilinear economy with an inverse demand function $p_d(q) = 5 - 0.4q$ and a single firm with a marginal cost $Cmg(q) = q$. The firm behaves as a monopolist. What is the monopoly price?

Exercise 1.11. Suppose that good q is produced by a monopolistic firm in the short run. If the market demand is given by $q^D = 100 - 0.5p$ and the firm's cost curve is $C(q) = 2q^2 + 10q + 4$,

a) Find the quantity produced, the price of the good, and the monopolist's profits.

b) Find the quantity produced, the price of the good, and the monopolist's profits if we consider a new cost curve given by $C(q) = 2q^2 + 10q$.

2 Price discrimination

Exercise 2.1. Describe the constraints of second degree price discrimination:

$$\begin{aligned} u_1(x_1) &\geq r_1 \\ u_1(x_1) - r_1 &\geq u_1(x_2) - r_2 \\ u_2(x_2) &\geq r_2 \\ u_2(x_2) - r_2 &\geq u_2(x_1) - r_1. \end{aligned}$$

Recall that the monopolist maximize

$$\Pi = (r_1 - cx_1) + (r_2 - cx_2).$$

Prove that

$$\begin{aligned} u_1'(x_1) - c + u_1'(x_1) - u_2'(x_1) &= 0 \\ u_2'(x_2) - c &= 0. \end{aligned}$$

Exercise 2.2. Prove that third degree price discrimination leads to (FOC)

$$p_1 \left[1 + \frac{dp_1}{dq_1}\frac{q_1}{p_1}\right] = c = p_2 \left[1 + \frac{dp_2}{dq_2}\frac{q_2}{p_2}\right].$$

Exercise 2.3. An economy has two types of consumers and two goods. The agent type Anakin has the following utility function:

$$u_A(x_{1A}, x_{2A}) = 4x_{1A} - \frac{x_{1A}^2}{3} + x_{2A}$$

and the agent type Ben Kenobi has the following utility function:

$$u_B(x_{1B}, x_{2B}) = 3x_{1B} - \frac{x_{1B}^2}{2} + x_{2B}.$$

Good 2 is the numeraire, and each consumer has an income of 100. Additionally, the economy has N consumers of both type Anakin and type Ben Kenobi.

a) Identify the type of consumer with high demand and the type with low demand for good x_1. Compare the marginal willingness to pay for each type of consumer for good x_1.

b) The monopolist produces good 1 with the following cost function $C(x_1) = cx_1$ and cannot discriminate prices. Find the optimal price and quantity of good x_1 that the monopolist will choose. For which values of c will the monopolist choose to sell to both types of consumers?

c) The monopolist engages in second-degree price discrimination by offering a menu of prices and quantities to each type of consumer (r_A, x_A) and (r_B, x_B). Based on this, formulate the monopolist's optimization problem and find the optimal values (r_A^*, x_A^*) and (r_B^*, x_B^*).

d) If the monopolist engages in third-degree price discrimination, what will be the prices and quantities set by the monopolist in the markets for Anakin-type and Ben Kenobi-type consumers?

e) If the monopolist engages in first-degree price discrimination, find the quantity produced by the monopolist in the market for good x. Calculate the consumer surplus and the monopolist's surplus.

Exercise 2.4. Consider a monopolist company that sells COVID-19 vaccines in two markets, the USA and Peru. In Peru, the inverse demand is given by:

$$p(x_{Pe}) = 8 - 2x_{Pe}$$

while in the USA:

$$p(x_{US}) = 10 - x_{US}.$$

The monopolist's cost function is $c(x_{Pe}, x_{US}) = x_{Pe} + x_{US}$, meaning the marginal cost of production is constant and equal to 1.

a) Which market has a higher willingness to pay (i.e., consumers are willing to pay more for a given quantity)?

b) Suppose the monopolist can discriminate prices between the two markets, meaning it can charge different prices in each market. Formulate the monopolist's profit maximization problem. Solve the problem. Calculate the profit-maximizing choices of x_{Pe}^D and x_{US}^D as well as the equilibrium prices p_{Pe}^D and p_{US}^D.

c) Compare the quantities and prices in the two markets. Interpret the results in relation to item a).

d) Now, suppose new legislation makes price discrimination between the two markets illegal. That is, the monopolist must charge the same price $p_{Pe}^M = p_{US}^M = p^M$ in both markets. Set up the new profit maximization problem for the firm.

e) Calculate the profit-maximizing choice of the total quantity produced X^M, the price p^M, and the quantities sold in each market x_{Pe}^M and x_{US}^M.

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when $p = p^c$, profits are zero and consumer surplus is $\int_0^{q^c} p(q)dq - p^c q^c$.

f) Compare the price p^M with the prices p^D_{Pe} and p^D_{US} under price discrimination. Make a similar comparison for the quantities produced with and without price discrimination.

Exercise 2.5. Medium difficulty, you must know how to solve first order differential equations. From Varian (1992). What shape must the demand curve have for $\dfrac{dp}{dc} = 1$? $MC = c$.

Exercise 2.6. From Varian (1992). Suppose that the inverse demand curve faced by a monopolist is given by $p(y,t)$, where t is a parameter that shifts the demand curve. For simplicity, assume that the monopolist has a technology with constant marginal costs. Derive an expression that shows how the production level responds to a variation in t. How does this expression simplify if $p(y,t) = a(y) + b(t)$?

Exercise 2.7. From Varian (1992). Suppose there are two consumers who can each purchase one unit of a good. If the quality of the good is q, consumer t obtains utility $u(q,t)$. The quality does not cost the monopolist anything. Suppose the maximum price that consumer t is willing to pay for quality q is given by w_t. The monopolist cannot distinguish between the two consumers, so they must offer at most two quantities from which the consumers can freely choose. Formulate the monopolist's profit maximization problem and analyze it thoroughly.

Exercise 2.8. From Varian (1992). There is a single monopolist whose technology exhibits constant marginal costs, i.e., $c(y) = cy$. The market demand curve has a constant elasticity ε. An *ad valorem* tax is imposed on the price of the good sold, so when the consumer pays the price p_d, the monopolist receives the price $p_s = (1-\tau)p_d$ (in this case, p_d is the demand price faced by the consumer, and p_s is the supply price faced by the producer). The tax authorities are considering the possibility of replacing the ad-valorem tax with a production tax t, in which case $p_d = p_s + t$. You have been hired to calculate the production tax t that is equivalent to the *ad valorem* tax τ, in the sense that the final price faced by the consumer is the same under both systems.

Exercise 2.9. Medium difficulty. From Varian (1992). Consider the case of a simple economy that behaves as if there were a single consumer whose utility function is $u_1(x_1)+u_2(x_2)+y$, where x_1 and x_2 are the quantities of goods 1 and 2, respectively, and y is the money available to spend on all other goods. Suppose that good 1 is supplied by a firm that acts competitively and good 2 by a firm that acts as a monopoly. The cost function for good i is $c_i(x_i)$, and there is a specific tax on the production of industry i in the amount of t_i. Assume that $c_i''(x_i) > 0$, $p_i''(x_i) < 0$, and $p_i'(x_i) < 0$.

a) Formulate the expressions corresponding to $\dfrac{dx_i}{dt_i}$ for $i = 1, 2$ and indicate their signs.

b) Given a variation in the quantities produced (dx_1, dx_2), formulate an expression for the change in welfare.

c) Suppose we are considering the possibility of imposing a tax on one of the two industries and using the revenues collected to subsidize the other. Determine which of the two firms should be taxed.

Exercise 2.10. Medium difficulty. A monopolist sells in two markets. The demand curve for their product is $x_1 = a_1 - b_1 p_1$ in market 1 and $x_2 = a_2 - b_2 p_2$ in market 2, where x_1 and x_2 are the quantities sold in each market, and p_1 and p_2 are the prices charged. The monopolist has zero marginal costs. Note that although the monopolist can charge different prices in the two markets, all units within a market must be sold at the same price.

a) Under what conditions relating to the parameters (a_1, b_1, a_2, b_2) will the monopolist make the optimal decision not to practice price discrimination? (Assume the solutions are interior.)

b) Now suppose that the demand functions take the form $x_i = A_i p_i^{-b_i}$, where $i = 1, 2$, and that the monopolist has a constant marginal cost $c > 0$. Under what conditions will the monopolist decide not to practice price discrimination?

3 Additional exercises in general equilibrium

3.1 Private Ownsership Economy

Exercise 3.1. Consider and economy with two consumers, D. Acemoglu (A) and R. Barro (B):

$$u_A(x_{A1}, x_{A2}) = \min\left\{x_{A1}, \frac{x_{A2}}{4}\right\}, \; \omega_A = (a, 1), \theta_A = 1/3$$
$$u_B(x_{B1}, x_{B2}) = (x_{B1})^{1/3}(x_{B2})^{2/3}, \; \omega_B = (1, b), \; \theta_B = 2/3,$$

with $a, b > 0$. Let

$$Y = \{(x_1, x_2) : 4x_2 + x_1 \leq 0, \; 4x_1 + x_2 \leq 0\}$$

be the firms technology.

a) Set the firm problem and solve it; specify all the price vector $p \in \Lambda$ for which the problem has a solution. Obtain the offer and profit correspondences.

b) Consider a specific $p \in \Lambda$. Set and solve the consumers problem (for each one).

c) Obtain the excess demand function and analyze if it satisfies the basic properties[2].

Exercise 3.2. Consider an economy with two goods, two consumers and a firm. Consumers have quasilinear utilities:

$$u_1(m_1, x_1) = m_1 + 4\ln x_1$$
$$u_2(m_2, x_2) = m_2 + \ln x_2.$$

Initial endowments are $\omega_1 = (100, 0)$ and $\omega_2 = (100, 0)$. Each one owns a fraction θ_i of a firm whose technology is given by

$$Y = \{(-m_e, x_e) : x_e = \sqrt{m_e}, \; x_e \geq 0, m_e \geq 0\}.$$

We take $x_i \geq 0$ but $m_i \in \mathbb{R}$. This is, consumers can consume a negative amount of m. Let p_m be the price of good m and p_x the price of good x.

1. Find the firm's offer.
2. Find each consumers demand.
3. Find the aggregated excess demand function.
4. There is a property which is not satisfied[3], which one? Why?
5. Can you normalize $p_m = 1$? Justify.
6. Prove that, in this economy, the equilibrium prices do not depend on the initial wealth[4] distribution.

[2] They are analogous to the Pure Exchange Economies case. See Mas-Colell et al. (1995) for a more detailed discussion.

[3] From the properties that aggregated demand function satisfy.

[4] Endowments and shares.

4 Advanced exercises

Exercise 4.1. Multi-product monopolist problem. Medium-difficulty. Consider now a monopolist producing $q_1, \cdots, q_n$ goods. He charges prices $p_1, \cdots, p_n$. Assume that his associated cost function is separable

$$C(q_1, \cdots, q_n) = \sum_{i=1}^{n} C_i(q_i).$$

Finally, let $D_i = D_i(p)$ the demand for good i.

1. Set the monopolist maximization problem.[5]

2. Prove that FOC provide
$$\left(D(p_i) + p_i \frac{\partial D_i}{\partial p_i} \right) + \sum_{j \neq i} p_j \frac{\partial D_j}{\partial p_i} - \sum_{j} \frac{\partial C}{\partial q_i} \frac{\partial D_j}{\partial p_i} = 0, \ \forall \ i.$$

3. Define $\varepsilon_{ii} = \frac{p_i}{D_i} \frac{\partial D_i}{\partial p_i}$ and $\varepsilon_{ij} = -\frac{\partial D_j}{\partial p_i} \frac{p_i}{D_j}$. Prove that
$$\frac{p_i - C_i'}{p_i} = \frac{1}{\varepsilon_{ii}} - \sum_{j \neq i} \frac{(p_j - C_j')D_j \varepsilon_{ij}}{R_i \varepsilon_{ii}}$$
where $R_i = p_i D_i$.

Exercise 4.2. Consider the problem of the monopolist in two periods

$$p_1 D(p_1) - C(D(p_1)) + \delta(p_2 D(p_2) - C(D(p_2))). \tag{3}$$

Set (3) as multi-product monopolist problem. Find $\frac{\partial D_1}{\partial p_2}$.

Exercise 4.3. Learning by doing. *In some industries, cost reductions are achieved over time simply because of learning. Learning by doing is especially apparent in industrial activity. This is for instance the case of the military aircraft production.* Consider a single-good monopolist producing at dates $t = 1, 2$. Assume that $q_t = D(p_t)$. The total cost at $t = 1$ is $C_1(q_1)$ and at $t = 2$, $C_2(q_1, q_2)$ where $\frac{\partial C_2}{\partial q_1} < 0$ (why?). Set the monopolist maximization problem as in (3), and prove that $p_1^* < p_m$. Interpret.

Exercise 4.4. You will use dynamic optimization in continous time. Assume that a monopolist has a unit-cost function such that $c = c(\omega(t))$ where $\omega(t)$ is the firm's experience at time t.

1. Explain why it is logical to assume that, denoting by $q = q(t)$ the output at time t, $\frac{d\omega}{dt} = q$.

2. Consider the monopolist maximization problem:
$$\max_{q(t) \in \mathbb{R}_+} \int_0^\infty [R(q(t)) - c(\omega(t))q(t)] e^{-rt} dt$$
$$\text{s. t. } w'(t) = q(t)$$
$$w(0) = w_0.$$

[5] Note that $C(q_1, \cdots, q_n) = C(D_1(p_1), \cdots, D_n(p_n))$.

Prove that

$$R'(q(t)) = c(\omega(t)) + \int_t^\infty c'(\omega(s))q(s)e^{-(s-t)}ds.$$

Hint: apply the Maximum Principle, see Acemoglu (2009) or Cerdá (2012). Note that the current value Hamiltonian is

$$\mathcal{H}(\omega(t), q(t), \psi(t), t) = R(q(t)) - c(\omega(t))q(t) + \psi(t)q(t).$$

Lima, September 17, 2024.

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