FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Solutions to the 1st Recitation

Microeconomics 2
Semester 2024-2

Professor: Pavel Coronado Castellanos
pavel.coronado@pucp.edu.pe
Teaching Assistants: Marcelo Gallardo & Fernanda Crousillat
marcelo.gallardo@pucp.edu.pe
https://marcelogallardob.github.io/
a20216775@pucp.edu.pe

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1 Selected Exercises

Exercise 1.1. Consider the classical utility maximization problem in $\mathbb{R}^n$

$$\begin{aligned} &\max\ u(\mathbf{x}) \\ &\text{s. t} : \mathbf{p}\cdot\mathbf{x} \leq I \\ &\qquad\quad \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Obtain the first order conditions associated to $\overline{L}$. You can start considering $n = 2$. This is $u(\mathbf{x}) = u(x_1, x_2)$. Assume $u(\cdot)$ is differentiable.

Solution: classical first order conditions lead to:

$$\frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i + \mu_i^* = 0, \quad i = 1, ..., n \tag{1}$$

$$\lambda^* \left(I - \sum_{i=1}^{n} p_i x_i^* \right) = 0 \tag{2}$$

$$I - \sum_{i=1}^{n} p_i x_i^* \geq 0 \tag{3}$$

$$\mu_i^* x_i^* = 0, \quad i = 1, ..., n \tag{4}$$

$$\lambda^*, \mu_1^*, ..., \mu_n^* \geq 0. \tag{5}$$

On the other hand, applying FOC to $\overline{L}$:

$$\frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i \leq 0, \quad i = 1, ..., n \tag{6}$$

$$I - \sum_{i=1}^{n} p_i x_i^* \geq 0 \tag{7}$$

$$x_i^* \left(\frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i \right) = 0, \quad i = 1, ..., n \tag{8}$$

$$\lambda^* \left(I - \sum_{i=1}^{n} p_i x_i^* \right) = 0. \tag{9}$$

Exercise 1.2. Solve the utility maximization problem for

$$u(x_1, x_2) = x_1 + x_2$$

in terms of p_1, p_2 and I. *Hint*: apply KKT theorem. Why can't you ensure that Lagrange is enough?

Solution: FOC for nonnegativity constraints lead to

$$\begin{aligned}
1 - \lambda^* p_1 &\leq 0 \\
1 - \lambda^* p_2 &\leq 0 \\
I - p_1 x_1^* - p_2 x_2^* &\geq 0 \\
x_1^*(1 - \lambda^* p_1) &= 0 \\
x_2^*(1 - \lambda^* p_2) &= 0 \\
\lambda^*(I - p_1 x_1^* - p_2 x_2^*) &= 0,
\end{aligned}$$

Where $\lambda^* \geq 0$. Now, since the utility function is strictly increasing in its arguments, we must have $I - p_1 x_1^* - p_2 x_2^* = 0$. However, we cannot conclude that x_1^* and x_2^* are strictly positive (unlike the case where u is, for example, of the Cobb-Douglas type). What we can affirm is that $x_1^*, x_2^* \neq 0$. Indeed, $x_1^* = x_2^* = 0$ implies that $I = 0$, which is a contradiction. Thus, we must analyze only three cases: $x_1^*, x_2^* > 0$, $x_1^* > 0, x_2^* = 0$, and $x_1^* = 0, x_2^* > 0$.

Note that in any of the three situations, the regularity condition holds, as the matrices

$$\begin{bmatrix} -p_1 & -p_2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -p_1 & -p_2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ -p_1 & -p_2 \end{bmatrix}$$

have full row rank.

Let's analyze each case individually. If $x_1^*, x_2^* > 0$,

$$\lambda^* = \frac{1}{p_1} = \frac{1}{p_2}.$$

This only makes sense if $p_1 = p_2 = p$. Thus, x_1^* and x_2^* are strictly positive only if $p_1 = p_2$. In that case,

$$\lambda^* = \frac{1}{p}, \ x_1^* \in \left] 0, \frac{I}{p} \right], \ x_2^* = \frac{I - p x_1^*}{p}. \tag{10}$$

Moreover, $u(x_1^*, x_2^*) = I/p$. Note that (10) implies that x_1^* and x_2^* can take an infinite number of values, which implies an infinite number of optimal bundles!

Now, suppose that $x_1^* > 0$ but $x_2^* = 0$. In this case, $x_1^* = I/p_1$ and $\lambda^* = 1/p_1$. Then, since

$$1 - \lambda^* p_2 = 1 - \frac{p_2}{p_1} \leq 0$$

we deduce that $p_1 \leq p_2$. Thus, when $p_1 \leq p_2$

$$(x_1^*, x_2^*, \lambda^*) = \left(\frac{I}{p_1}, 0, \frac{1}{p_1}\right). \tag{11}$$

Finally, if $x_2^* > 0$ but $x_1^* = 0$, the situation is similar to case 2, and we have that, when $p_2 \geq p_1$

$$(x_1^*, x_2^*, \lambda^*) = \left(0, \frac{I}{p_2}, \frac{1}{p_2}\right). \tag{12}$$

In summary, there are 3 possible situations:

1. $p_2 < p_1$: $(x_1^*, x_2^*) = \left(\frac{I}{p_1}, 0\right)$
2. $p_1 > p_2$: $(x_1^*, x_2^*) = \left(0, \frac{I}{p_2}\right)$
3. $p_1 = p_2$: Combining (10) with (11) and (11), we deduce that any point (x_1^*, x_2^*) on the budget line generates the same utility, and therefore, any point on the budget line constitutes a solution to the problem.

Let us briefly interpret the results. If $p_1 > p_2$, it is more expensive to consume good 1. Therefore, the consumer will substitute[1] all consumption of good 1 by consuming only good 2. Conversely, if $p_2 > p_1$, they will do the opposite, consuming only good 1. Finally, in the case $p_1 = p_2 = p$, they will consume any convex combination of the points $\left(\frac{I}{p}, 0\right)$ and $\left(0, \frac{I}{p}\right)$

$$(x_1^*, x_2^*) = \theta\left(\frac{I}{p}, 0\right) + (1-\theta)\left(0, \frac{I}{p}\right), \ \theta \in [0,1],$$

since they generate the same utility.

Exercise 1.3. Solve the following maximization problem,

$$\begin{aligned} &\max\ x_1 x_2 \\ &\text{s. t. } x_1 + x_2^2 \leq 2 \\ &\qquad x_1, x_2 \geq 0. \end{aligned}$$

Solution: first, since the feasible region is bounded and closed (hence compact), and the objective function is continuous, Weierstrass theorem ensures the existence to a solution.

[1]Indeed, the linear utility function is also known as a utility of perfect substitutes.

Then, non negativity first order constraints lead to (we omit $*$):

$$\begin{aligned}
\frac{\partial \overline{L}}{\partial x_1} &= x_2 - \lambda \leq 0 \\
\frac{\partial \overline{L}}{\partial x_2} &= x_1 - 2\lambda x_2 \leq 0 \\
x_1 \frac{\partial \overline{L}}{\partial x_1} &= x_1(x_2 - \lambda) = 0 \\
x_2 \frac{\partial \overline{L}}{\partial x_2} &= x_2(x_1 - 2\lambda x_2) = 0 \\
x_1 \frac{\partial \overline{L}}{\partial \lambda} &= x_1(2 - x_1 - x_2^2) = 0 \\
\lambda \frac{\partial \overline{L}}{\partial x \lambda} &= \lambda(2 - x_1 - x_2^2) = 0.
\end{aligned}$$

If $\lambda = 0$, then $x_1 x_2 = 0$, but, this is not optimal since f takes positive values. Thus, $\lambda > 0$. It follows that

$$\begin{aligned}
2 - x_1 - x_2^2 &= 0 \\
x_2 &= \lambda.
\end{aligned}$$

Moreover, since $x_2 = \lambda$,

$$\begin{aligned}
2 - x_1 - x_2^2 &= 0 \\
x_1 - 2x_2^2 &= 0.
\end{aligned}$$

The solutions to this pair of nonlinear equations are

$$(x_1^*, x_2^*) = \begin{cases} \left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right) \\ \left(\frac{4}{3}, -\sqrt{\frac{2}{3}}\right) \end{cases}$$

Finally, since $\lambda \geq 0$, the solution is

$$\left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right).$$

Exercise 1.4. Medium difficulty. Consider the following optimization problem,

$$\begin{aligned}
\min \quad & -\sum_{i=1}^{n} \ln(\alpha_i + x_i) \\
\text{s. t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0,
\end{aligned}$$

where $\alpha_i > 0$ are parameters. Solve this problem applying KKT (non negativity constraints).

Solution: note first that this is a mixed problem since we have equality and inequality constraints. You can consult the bibliography for a more detailed analysis on this subject. Nonetheless, what you can do, is to create artificially the constraints $\sum_i x_i \leq 1$ and $\sum_i x_i \geq 1$, which combined lead to $\sum_i x_i = 1$. Now, let us solve this problem using the classical Lagrangian. It is your homework to proceed via $\overline{L}$ and obtain the same results. Hence, let

$$L(x, \lambda, \mu) = -\sum_{i=1}^{n} \ln(\alpha_i + x_i) + \lambda \left(\sum_{i=1}^{n} x_i - 1 \right) - \sum_{i=1}^{n} \mu_i x_i.$$

KKT FOC provide

$$-\frac{1}{\alpha_i + x_i} - \mu_i + \lambda = 0$$
$$\sum_{i=1}^{n} x_i = 1$$
$$\mu_i x_i = 0.$$

If, for some i, $\mu_i = 0$, then

$$x_i = \frac{1}{\lambda} - \alpha_i.$$

Note that $\lambda \geq 0$ since $\lambda = \frac{1}{\alpha_i + x_i}$ for all i. Now, $x_i \geq 0$ iff $\lambda \leq \frac{1}{\alpha_i}$. If $x_i = 0$, then

$$\mu_i = -\frac{1}{\alpha_i} + \lambda,$$

which is positive iff $\lambda \geq 1/\alpha_i$. Now, if $\lambda \geq 1/\alpha_i$, x_i cannot be positive since, if it was the case,

$$\mu_i = \lambda - \frac{1}{\alpha_i + x_i} > 0.$$

Which violates $\mu_i x_i = 0$. Thus, if $x_i > 0$, $\lambda \in (0, 1/\alpha_i)$. Therefore,

$$x_i = \max\{\lambda - 1/\alpha_i, 0\}.$$

Finally, s

$$\sum_i x_i = \sum_i \max\{\lambda - 1/\alpha_i, 0\} = 1.$$

There is no closed formula for λ but, we can ensure its existence since $\sum_i \max\{\theta - \alpha_i, 0\} = g(\theta)$ is piecewise linear, continuous and increasing, with breakpoints α_i.

Exercise 1.5. Formulate the respective optimization problems (derive KKT first order conditions):

1. Expenditure minimization problem.

2. Profit maximization problem.

3. Cost minimization problem.

In each case, assume differentiability.

Solution: the expenditure minimization problem is

$$\begin{aligned} \min \ & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.}: \ & u(x_1, \cdots, x_n) \geq \overline{u} \\ & x_1, \cdots, x_n \geq 0. \end{aligned}$$

Note that this is equivalent to

$$\begin{aligned} \max \ & -\sum_{i=1}^{n} p_i x_i \\ \text{s.t.}: \ & -u(x_1, \cdots, x_n) \leq -\overline{u} \\ & -x_1, \cdots, -x_n \leq 0. \end{aligned}$$

$$L(x, \lambda, \mu) = -\sum_{i=1}^{n} p_i x_i + \lambda(-\overline{u} + \underbrace{u(x_1, ..., x_n)}_{=u(x)}) - \sum_{i=1}^{n} \mu_i x_i.$$

Hence, FOC provide

$$\begin{aligned} -p_i + \lambda \frac{\partial u}{\partial x_i} - \mu_i &= 0 \\ \mu_i x_i &= 0 \\ \mu_i &\geq 0 \\ \lambda(u(x) - \overline{u}) &= 0 \\ -u(x) &\leq \overline{u}. \end{aligned}$$

Note that, in this case, FOC appled to $\overline{L} = -p \cdot x + \lambda(u(x) - \overline{u})$ are

$$\begin{aligned} p_i - \lambda \frac{\partial u}{\partial x_i} &\geq 0 \\ x_i \left(p_i - \lambda \frac{\partial u}{\partial x_i} \right) &= 0 \\ \lambda(u(x) - \overline{u}) &= 0 \\ u(x) &\geq \overline{u}. \end{aligned}$$

With respect to the profit maximization problem

$$\max_{x \geq 0} \ pf(x_1, \cdots, x_n) - \sum_{i=1}^{n} p_i x_i,$$

FOC provide

$$p \frac{\partial f}{\partial x_i} - w_i - \mu_i = 0.$$

$$p \frac{\partial f}{\partial x_i} - w_i - \mu_i = 0.$$

Hence,

$$p\frac{\partial f}{\partial x_i} - w_i \geq 0.$$

Finally, cost minimization problem, which is very similar to the expenditure minimization problem,

$$\begin{aligned} &\min \sum_{i=1}^{n} w_i x_i \\ &\text{s.t. } f(x_1, \cdots, x_n) \geq q \\ &\quad\; x_1, \cdots, x_n \geq 0 \end{aligned}$$

leads to the same formulation just changing w_i by p_i, $f(x)$ by $u(x)$ and $\overline{u}$ by q.

Further references at Columbia Lecture Notes.

2 Additional exercises

We provide hints for some of the additional exercises.

Exercise 2.1. Medium difficulty. With respect to the utility maximization problem, explain why Inada conditions, given below, ensure that it can be solved by Lagrange.

1. $u(\mathbf{0}) = 0$
2. u differentiable and concave
3. $\partial u(\mathbf{x}^*)/\partial x_i > 0 \;\; \forall\, i = 1, \ldots, n$
4. $\lim_{x_i \to 0^+} \partial u(\mathbf{x})/\partial x_i = \infty, \;\forall\, i = 1, ..., n$
5. $\lim_{x_i \to \infty} \partial u(\mathbf{x})/\partial x_i = 0, \;\forall\, i = 1, ..., n.$

Hint: condition 4 is key.

Since $\lim_{x_i \to 0^+} \partial u(\mathbf{x})/\partial x_i = \infty, \;\forall\, i = 1, ..., n$, you can prove, using the definition of derivative as a limit, that $x_i^* > 0$. Hence, Lagrange can be applied since the associated $\mu_i^* = 0$. Differentiability is key and the condition $u(0) = 0$ is used for the EMP.

Exercise 2.2. Solve the following optimization (utility maximization) problems:

1. $\max \; x_1 x_2$ s.t. $x_1 + x_2 \leq 1$, $x_1, x_2 \geq 0$.
2. $\max \; \ln x_1 + \ln x_2$ s.t. $2x_1 + 3x_2 \leq 5$, $x_1, x_2 > 0$.
3. $\max \; \min\{x_1, 2x_2\}$ s.t. $x_1 + x_2 \leq 2$.

Hint: argue why you can apply Lagrange instead of KKT. In the last one, you can't apply neither Lagrange or KKT, why?

You can apply in both cases (1, 2) Lagrange since both utility functions satisfy Inada conditions. The first one is a Cobb-Douglas $x^a y^b$. It is well known that the solution is

$$x^* = \frac{aI}{p_x(a+b)}$$
$$y^* = \frac{bI}{p_y(a+b)}.$$

Hence,

$$x_1^* = \frac{1}{2}$$
$$y^* = \frac{1}{2}.$$

In the second case, since $\ln x_1 + \ln x_2$ is equivalent to $x_1 x_2$ (strictly monotone transformation of the utility), the solution is

$$x_1^* = \frac{1}{2\cdot 2} = \frac{1}{4}$$
$$y^* = \frac{1}{2\cdot 3} = \frac{1}{6}.$$

Finally, since $\min\{x_1, x_2\}$ is not differentiable, the procedure is ad-hoc. At the optimum, $x_1 = 2x_2$. Hence, since Leontief preference is monotone,

$$2x_2 + x_2 = 2 \implies x_2^* = \frac{2}{3} \implies x_1^* = \frac{4}{3}.$$

Exercise 2.3. Medium difficulty. Thomas Sargent (Tom) has the following utility function:

$$u(\mathbf{x}) = \prod_{i=1}^{n} x_i^{\alpha_i},\ 0 < \alpha_i < 1,\ \sum_{i=1}^{n} \alpha_i = 1.$$

Solve Tom's maximization problem considering $\mathbf{p} \in \mathbb{R}^n_{++}$ and $I > 0$. Obtain the Marshallian demands for each good consumed by Tom and verify Roy's identity.

Applying Largange you can check that the solution is

$$x_i^* = \frac{I\alpha_i}{p_i \sum_{i=1}^{n} \alpha_i} = \frac{\alpha_i I}{p_i}.$$

Exercise 2.4. High-difficulty, not assessable exercise. Requires some elements from Microeconomic I and the Enveloppe Theorem. Tirole's expenditure function is given by:

$$e(\mathbf{p}, \overline{u}) = \exp\left\{ \sum_{\ell=1}^{L} \alpha_\ell \ln(p_\ell) + \left(\prod_{\ell=1}^{L} p_\ell^{\beta_\ell} \right) \overline{u} \right\},\ \mathbf{p} \in \mathbb{R}^L_{++}.$$

Assume (this is known as the duality theorem) that $e(\mathbf{p}, V(\mathbf{p}, I)) = I$, where I is the income in the utility maximization problem and V is the indirect utility function. Derive Tirole's indirect utility function and verify Roy's identity. Impose any conditions you deem appropriate on the parameter vector $(\boldsymbol{\alpha}, \boldsymbol{\beta})$[2]. *Hint*: you should find that $\beta_\ell = 0$ for every ℓ and that $\sum_\ell \alpha_\ell = 1$.

[2]Recall that expenditure functions are concave with respect to prices, non-decreasing in p_ℓ, and

We have that

$$V(p,I) = \frac{\ln I - \sum_{\ell=1}^{L} \alpha_\ell \ln p_\ell}{\prod_{\ell=1}^{L} p_\ell^{\beta_\ell}}.$$

$$V(p,I) = \ln I - \sum_{\ell=1}^{L} \alpha_\ell \ln p_\ell = \ln\left(\prod_{\ell=1}^{L} \frac{I^{1/L}}{p_\ell^{\alpha}}\right)$$

Now, to simplify the work, given that e must satisfy the aforementioned properties (homogeneity of degree 1, increasing in prices, strictly increasing in the utility level), $\beta_\ell = 0$ for all ℓ and $\sum_{\ell=1}^{L} \alpha_\ell = 1$. Indeed,

$$\frac{\partial e}{\partial p_i} = \frac{e(p,u)}{p_i}\left(\alpha_i + u\beta_i \prod_{\ell=1}^{L} p_\ell^{\beta_\ell}\right).$$

Since this derivative must be positive, $\alpha_i, \beta_i \geq 0$. Then,

$$e(\lambda p, \overline{u}) = \lambda^{\sum_\ell \alpha_\ell} \exp\left\{\sum_\ell \alpha_\ell p_\ell + \lambda^{\sum_\ell \beta_\ell} \overline{u} \prod_{\ell=1}^{L} p_\ell^{\beta_\ell}\right\}.$$

This must hold for all p and $\overline{u}$, in particular $p = (1, \cdots, 1)$ and $\overline{u} = 1$. Thus,

$$\begin{aligned} \ln e(p,\overline{u}) &= \left(\sum_\ell \alpha_\ell\right) \ln \lambda + \lambda^{\sum_\ell \beta_\ell} \\ \ln \lambda e(p,\overline{u}) &= \ln \lambda + 1. \end{aligned}$$

In this way, $\sum_\ell \alpha_\ell = 1$ and $\sum_\ell \beta_\ell = 0$. Then, $\beta_\ell = 0$ for all ℓ. We conclude then, applying Roy's identity and replacing with the parameters, that the ordinary demand in Tirole is

$$x^* = \left(\frac{I}{\alpha_1}, \cdots, \frac{I}{\alpha_L}\right).$$

Exercise 2.5. Medium-difficulty, not assessable exercise. Requires Cramer rule and differentiation. Consider the utility maximization problem with $p_1, p_2, I > 0$ and $u \in C^2(\mathbb{R}^2)$. Additionally, assume that $\dfrac{\partial^2 u}{\partial x_i^2} < 0$, $\dfrac{\partial u}{\partial x_i} > 0$, and $\dfrac{\partial^2 u}{\partial x_1 \partial x_2} > 0, i = 1, 2$. Assume that $\mathbf{x}^* \in \mathbb{R}^2_{++}$ satisfies the Lagrange equations. Using the method of differentials (comparative statics), determine the effect (whether positive, negative, or inconclusive) of $\dfrac{\partial x_2^*}{\partial I}$, where (x_1^*, x_2^*) is the solution to the utility maximization problem considered. Provide an interpretation.

The Lagrange equations at the (or an) optimal point provide

$$\begin{aligned} u_{x_1} - \lambda p_1 &= 0 \\ u_{x_2} - \lambda p_2 &= 0 \\ I - p_1 x_1 - p_2 x_2 &= 0. \end{aligned}$$

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increasing in $\overline{u}$.

Differentiating,

$$
\begin{aligned}
u_{x_1x_1}dx_1 + u_{x_1x_2}dx_2 - d\lambda p_1 - \lambda dp_1 &= 0 \\
u_{x_2x_2}dx_2 + u_{x_1x_2}dx_1 - d\lambda p_2 - \lambda dp_2 &= 0 \\
dI - dp_1x_1 - p_1dx_1 - dp_2x_2 - p_2dx_2 &= 0.
\end{aligned}
$$

Then,

$$
\underbrace{\begin{bmatrix} u_{x_1x_1} & u_{x_1x_2} & -p_1 \\ u_{x_1x_2} & u_{x_2x_2} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}}_{=A} \begin{bmatrix} dx_1 \\ dx_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ x_1dp_1 + x_2dp_2 - dI \end{bmatrix}.
$$

Applying Cramer's rule and canceling the effects that are not of interest,

$$
dx_1 = \frac{1}{|A|} \begin{vmatrix} 0 & u_{x_1x_2} & -p_1 \\ 0 & u_{x_2x_2} & -p_2 \\ -dI & -p_2 & 0 \end{vmatrix} = \frac{u_{x_1x_2}p_2 - p_1u_{x_2x_2}}{|A|}dI > 0.
$$

The assumptions are those that ensure $\frac{u_{x_1x_2}p_2 - p_1u_{x_2x_2}}{|A|} > 0$. Thus, $\frac{dx_1}{dI} > 0$ (as expected: income effect).