FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Recitation 1

Microeconomics 2
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Professor: Pavel Coronado Castellanos
pavel.coronado@pucp.edu.pe
Teaching Assistants: Marcelo Gallardo & Fernanda Crousillat
marcelo.gallardo@pucp.edu.pe
https://marcelogallardob.github.io/
a20216775@pucp.edu.pe

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1 Karush-Kuhn-Tucker

In this first recitation, we will review much of the theory of Karush-Kuhn-Tucker optimization theory in $\mathbb{R}^n$. The main bibliographic sources for these topics are de la Fuente (2000), Sundaram (1996) and Chavez and Gallardo (2024) (Chapter 9). This is just a review; a thorough and detailed treatment can be found in these texts. For Lagrange optimization theory and classical static unconstrained optimization theory, see Simon and Blume (1994).

Consider the nonlinear programming problem:

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n} \ & f(\mathbf{x}) \\ \text{s.t. } & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \ldots, p. \end{aligned} \tag{1}$$

Assume that the functions f, g_i for $i = 1, \ldots, m$, and h_j for $j = 1, \ldots, p$ are continuously differentiable. If $\mathbf{x}^*$ is a local minimum and the gradients of the active constraints $g_i(\mathbf{x}^*) = 0$ and the equality constraints $h_j(\mathbf{x}^*) = 0$ are linearly independent, then there exist Lagrange multipliers $\lambda_i^* \geq 0$ for $i = 1, \ldots, m$, and μ_j^* for $j = 1, \ldots, p$ such that:

$$\begin{aligned} \nabla f(\mathbf{x}^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^p \mu_j^* \nabla h_j(\mathbf{x}^*) = \mathbf{0} \\ \lambda_i^* g_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, m \\ \lambda_i^* \geq 0, \quad g_i(\mathbf{x}^*) \leq 0, \quad i = 1, \ldots, m \\ h_j(\mathbf{x}^*) = 0, \quad j = 1, \ldots, p \end{aligned} \tag{2}$$

A particular case is

$$\mathcal{P}:\begin{cases}\max & f(\mathbf{x})\\ \text{s. t}: & g(\mathbf{x})\le \mathbf{b}\\ & \quad\mathbf{x}\ge \mathbf{0},\end{cases}$$

known as Karush-Kuhn-Tucker problem with non-negativity constraints. The associated Lagrangian function is

$$L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{b}-g(\mathbf{x})) + \boldsymbol{\mu}\mathbf{x}.$$

First order conditions state that there exist $\boldsymbol{\lambda}^*\in\mathbb{R}^m$ and $\boldsymbol{\mu}^*\in\mathbb{R}^n$ such that

$$\frac{\partial L(\mathbf{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} - \sum_{j=1}^{m}\lambda_j^*\frac{\partial g_j(\mathbf{x}^*)}{\partial x_i} + \mu_i^* = 0,\ \ i=1,...,n \tag{3}$$

$$\lambda_j^*(b_j - g_j(\mathbf{x}^*)) = 0, \quad j=1,...,m \tag{4}$$

$$b_j - g_j(\mathbf{x}^*) \ge 0, \quad j=1,...,m \tag{5}$$

$$\mu_i^* x_i^* = 0, \quad i=1,...,n \tag{6}$$

$$\lambda_j^*, \mu_i^* \ge 0, \quad i=1,...,n;\ j=1,...,m. \tag{7}$$

From these conditions, an alternative formulation of the necessary conditions can be obtained, which is sometimes easier to work with. To do this, let's first define

$$\overline{L}(\mathbf{x},\boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{b}-g(\mathbf{x})).$$

That is,

$$L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = \overline{L}(\mathbf{x},\boldsymbol{\lambda}) + \boldsymbol{\mu}\mathbf{x}.$$

Since for each $i=1,\dots,n$, we obtain

$$\frac{\partial L(\mathbf{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)}{\partial x_i} = \frac{\partial \overline{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*)}{\partial x_i} + \mu_i^* = 0,$$

and since $\mu_i^*\ge 0$, then

$$\frac{\partial \overline{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*)}{\partial x_i} = -\mu_i^* \le 0.$$

Also taking into account that $\mu_i^* x_i^* = 0$, it follows that

$$x_i^*\frac{\partial \overline{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*)}{\partial x_i} = 0, \quad i=1,\dots,n.$$

On the other hand, for each $j=1,\dots,m$, from the definition of L and (4), we obtain

$$\frac{\partial \overline{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*)}{\partial \lambda_j} = b_j - g_j(\mathbf{x}^*) \ge 0$$

$$\lambda_j^*\frac{\partial \overline{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*)}{\partial \lambda_j} = \lambda_j^*(b_j - g_j(\mathbf{x}^*) = 0.$$

Thus, we obtain the following necessary conditions relative to $\overline{L}$:

$$\frac{\partial \overline{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} \leq 0, \quad i = 1, ..., n \tag{8}$$

$$\frac{\partial \overline{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial \lambda_j} \geq 0, \quad j = 1, ..., m \tag{9}$$

$$x_i^* \frac{\partial \overline{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} = 0, \quad i = 1, ..., n \tag{10}$$

$$\lambda_j^* \frac{\partial \overline{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial \lambda_j} = 0, \quad j = 1, ..., m \tag{11}$$

$$\lambda_j^* \geq 0, \quad j = 1, ..., m. \tag{12}$$

The advantage of this formulation in terms of $\overline{L}$ is that we have $n + m$ variables instead of $2n + m$ variables (formulation using L). Thus, the point $\mathbf{x}^*$ must satisfy these new conditions for certain $\lambda_1^*, ..., \lambda_m^*$, which are unique and non-negative.

1.1 Second-Order Conditions for Optimality

For a solution $\mathbf{x}^*$ to be a local maximum or minimum, the second-order conditions must hold as well. Conditions for a minimum are analogous.

1.1.1 Second-Order Sufficient Conditions (SOSC) for Lagrange

If $\mathbf{x}^*$ is a solution to the Lagrange problem and the first-order conditions hold, then $\mathbf{x}^*$ is a local maximum if:

$$\mathbf{d}^T \nabla^2 L(\mathbf{x}^*, \boldsymbol{\mu}^*) \mathbf{d} < 0 \tag{13}$$

for all $\mathbf{d} \neq \mathbf{0}$ such that $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0$ for all j.

1.1.2 Second-Order Necessary Conditions (SONC) for KKT

If x^* is a local maximum for the KKT problem and the first-order conditions hold, then it is necessary that:

$$\mathbf{d}^T \nabla^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{d} \leq 0 \tag{14}$$

for all d that satisfy the linearized constraints $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} = 0$ for active constraints and $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0$ for all j.

For practical examples related to optimization problems with equality or inequality constraints, refer to Simon and Blume (1994). Next, we proceed with the exercises, which are the focus of this document. Theoretical aspects of Microeconomic Theory which are not covered in this section can be found in Echenique (2015), Varian (1992), Mas-Colell et al. (1995) or Jehle and Reny (2011).

1.2 Selected exercises

Exercise 1.1. Consider the classical utility maximization problem in $\mathbb{R}^n$

$$\begin{aligned} &\max\ u(\mathbf{x}) \\ &\text{s. t}: \mathbf{p}\cdot\mathbf{x} \leq I \\ &\qquad\quad \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Obtain the first order conditions associated to $\overline{L}$. You can start considering $n = 2$. This is $u(\mathbf{x}) = u(x_1, x_2)$. Assume $u(\cdot)$ is differentiable.

Exercise 1.2. Solve the utility maximization problem for

$$u(x_1, x_2) = x_1 + x_2$$

in terms of p_1, p_2 and I. *Hint*: apply KKT theorem. Why can't you ensure that Lagrange is enough?

Exercise 1.3. Solve the following maximization problem,

$$\begin{aligned} &\max\ x_1 x_2 \\ &\text{s. t. } x_1 + x_2^2 \leq 2 \\ &\qquad x_1, x_2 \geq 0. \end{aligned}$$

Exercise 1.4. Medium difficulty. Consider the following optimization problem,

$$\begin{aligned} &\min\ -\sum_{i=1}^{n} \ln(\alpha_i + x_i) \\ &\text{s. t. } \sum_{i=1}^{n} x_i = 1 \\ &\qquad x_i \geq 0, \end{aligned}$$

where $\alpha_i > 0$ are parameters. Solve this problem applying KKT (non negativity constraints).

Exercise 1.5. Formulate the respective optimization problems (derive KKT first order conditions):

1. Expenditure minimization problem.

2. Profit maximization problem.

3. Cost minimization problem.

In each case, assume differentiability.

1.3 Additional problems: Lagrange

Exercise 1.6. Medium difficulty. With respect to the utility maximization problem, explain why Inada conditions, given below, ensure that it can be solved by Lagrange.

1. $u(\mathbf{0}) = 0$
2. u differentiable and concave
3. $\partial u(\mathbf{x}^*)/\partial x_i > 0 \ \ \forall\, i = 1, \ldots, n$
4. $\lim_{x_i \to 0^+} \partial u(\mathbf{x})/\partial x_i = \infty, \ \forall\, i = 1, ..., n$
5. $\lim_{x_i \to \infty} \partial u(\mathbf{x})/\partial x_i = 0, \ \forall\, i = 1, ..., n.$

Hint: condition 4 is key.

Exercise 1.7. Solve the following optimization (utility maximization) problems:

1. max $x_1 x_2$ s.t. $x_1 + x_2 \leq 1$, $x_1, x_2 \geq 0$.
2. max $\ln x_1 + \ln x_2$ s.t. $2x_1 + 3x_2 \leq 5$, $x_1, x_2 > 0$.
3. max $\min\{x_1, 2x_2\}$ s.t. $x_1 + x_2 \leq 2$.

Hint: argue why you can apply Lagrange instead of KKT. In the last one, you can't apply neither Lagrange or KKT, why?

Exercise 1.8. Medium difficulty. Thomas Sargent (Tom) has the following utility function:

$$u(\mathbf{x}) = \prod_{i=1}^{n} x_i^{\alpha_i}, \ 0 < \alpha_i < 1, \ \sum_{i=1}^{n} \alpha_i = 1.$$

Solve Tom's maximization problem considering $\mathbf{p} \in \mathbb{R}^n_{++}$ and $I > 0$. Obtain the Marshallian demands for each good consumed by Tom and verify Roy's identity.

Exercise 1.9. Formulate the utility maximization problem for a Stone-Geary utility function. Derive first order conditions and argue why you can apply Lagrange instead of KKT.

Exercise 1.10. Let $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, with $\alpha, \beta \in [0, 1]$. Show that

$$c(w_1, w_2, q) = q^{\frac{1}{\alpha+\beta}} \theta \phi(w_1, w_2)$$

where $\phi(w_1, w_2) = w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}}$, $q > 0$ is the production level and

$$\theta = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Note that $c(w_1, w_2, q)$ is the cost function.

1.4 Supplementary and advanced exercises

Exercise 1.11. Formulate KKT first order conditions for a constrained minimization problem with inequality constraints.

More exercises concerning KKT problems in: Applications of Lagrangian: Kuhn Tucker Conditions and NMSA403 Optimization Theory – Exercises.

The following problems are designed to refresh your understanding of some important aspects of consumer theory. Please refer yourself to Mas-Colell et al. (1995) and Chavez and Gallardo (2024) for a deeper understanding.

Exercise 1.12. High-difficulty, not assessable exercise. Requires some elements from Microeconomic I and the Enveloppe Theorem. Tirole's expenditure function is given by:

$$e(\mathbf{p}, \overline{u}) = \exp\left\{\sum_{\ell=1}^{L} \alpha_\ell \ln(p_\ell) + \left(\prod_{\ell=1}^{L} p_\ell^{\beta_\ell}\right) \overline{u}\right\}, \ \mathbf{p} \in \mathbb{R}^L_{++}.$$

Assume (this is known as the duality theorem) that $e(\mathbf{p}, V(\mathbf{p}, I)) = I$, where I is the income in the utility maximization problem and V is the indirect utility function. Derive Tirole's indirect utility function and verify Roy's identity. Impose any conditions you deem appropriate on the parameter vector $(\boldsymbol{\alpha}, \boldsymbol{\beta})$[1]. *Hint*: you should find that $\beta_\ell = 0$ for every ℓ and that $\sum_\ell \alpha_\ell = 1$.

Exercise 1.13. Requires Microeconomics I and Mathematics for Economists IV. Daron Acemoglu has preferences represented by $u(x_1, x_2) = (x_1 + 1)(x_2 + 1)$. Prove that Acemoglu has convex preferences. Are they strictly convex? Perform the same analysis for the preferences of Robert Barro, represented by $v(x_1, x_2) = \min\left\{\frac{x_1}{3}, \frac{x_2}{10}\right\}$. Analyze if the preferences of Acemoglu and Barro are: monotone, locally non satiated and continous.

Exercise 1.14. Medium-difficulty, not assessable exercise. Requires Cramer rule and differentiation. Consider the utility maximization problem with $p_1, p_2, I > 0$ and $u \in C^2(\mathbb{R}^2)$. Additionally, assume that $\dfrac{\partial^2 u}{\partial x_i^2} < 0$, $\dfrac{\partial u}{\partial x_i} > 0$, and $\dfrac{\partial^2 u}{\partial x_1 \partial x_2} > 0, i = 1, 2$. Assume that $\mathbf{x}^* \in \mathbb{R}^2_{++}$ satisfies the Lagrange equations. Using the method of differentials (comparative statics), determine the effect (whether positive, negative, or inconclusive) of $\dfrac{\partial x_2^*}{\partial I}$, where (x_1^*, x_2^*) is the solution to the utility maximization problem considered. Provide an interpretation.

Exercise 1.15. Medium to hard difficulty. Let $n \geq 2$. Consider the following problem: min x_1 s.t. $\sum_{i=1}^n \left(x_i - \frac{1}{n}\right)^2 \leq \frac{1}{n(n-1)}$ and $\sum_{i=1}^n x_i = 1$. Prove that

$$(\mathbf{x}^*, \lambda^*, \mu^*) = \left(0, \frac{1}{n-1}, \cdots, \frac{1}{n-1}, -\frac{1}{n}, \frac{n-1}{2}\right).$$

Lima, August 19, 2024.

[1] Recall that expenditure functions are concave with respect to prices, non-decreasing in p_ℓ, and

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increasing in $\overline{u}$.