Lecture Notes in General Equilibrium

Marcelo Gallardo

2024

Pontificia Universidad Católica del Perú

marcelo.gallardo@pucp.edu.pe

This document is based on numerous sources. Its fundamental structure is derived from the «Advanced Microeconomics, Module 1» course taught by Professor Alejandro Lugón at PUCP, and the lecture notes of the course Microeconomics 2, taught by professor Pavel Coronado at PUCP. Additionally, the lecture notes from Professor Federico Echenique (University of California, Berkeley), and those from Professor Jonathan Levin (Stanford University) have been crucial. Although the notation is not identical, the approach largely follows Chapter 10 of the pre-published book Linear Algebra and Optimization for Economic Analysis by Chávez and Gallardo (2024). Finally, «Microeconomic Theory» by Mas-Colell et al., along with «Existence and Optimality of Competitive Equilibrium», have been key foundational sources. Preliminaries to this document can be found in F. Echenique's lecture notes or in this recap of Consumer and Producer Theory. We start with the more basic models: 2×2 economy and Robinson Crusoe's economy. Then, we study the 2×2 production model. Once this is concluded, we pass to the general situation: pure exchange economies and private ownership economies. We state both welfare theorems (we don't prove them but we provide the references). We provide the proof of the existence of Walrasian equilibrium in the case of functions and discuss the issue of uniqueness. Finally, we address the topic of aggregation. I will use $>$ or $>>$ for strict inequality in each entry.

Contents

1 Economy 2×2 **3**

1.1 Exercises 9

2 Robinson Crusoe Economy **11**

2.1 Exercises 13

3 The 2×2 production model **15**

4 Pure Exchange Economy **20**

4.1 Efficiency 21

4.2 Excess of demand and existence of the Walrasian equilibrium 23

4.3 Exercises 26

5 Economies with production **29**

5.1 Private ownership economies 31

5.2 Exercises 35

6 Uniqueness of the Walrasian equilibrium **38**

7 The core **41**

8 Welfare and aggregation **45**

8.1 Negishi's Method 48

9 Conclusion **50**

1 Economy 2×2

We consider an economy with 2 consumers and 2 goods without the possibility of transformation (production). Each consumer possesses initial quantities of each good; these are (initial) endowments:

$$\omega^1 = (\omega_1^1, \omega_2^1)$$
$$\omega^2 = (\omega_1^2, \omega_2^2).$$

Exchange is voluntary and occurs when both benefit from it. To measure when they benefit, we associate to the preferences of the consumers $\succeq_i$ utility functions $u_i(x_1^i, x_2^i)$, $i = 1, 2$. Thus, each consumer is defined by their pair (u^i, ω^i).[1]

Remark. A very useful graphical tool in studying this economy is the well-known Edgeworth box. It is a rectangle $\Box$ with length $\overline{\omega}_1 = \omega_1^1 + \omega_1^2$ and height $\overline{\omega}_2 = \omega_2^1 + \omega_2^2$. The points in $\Box$ represent all possible ways of distributing the goods. In the Edgeworth box, it is also possible to represent the indifference curves of the consumers.

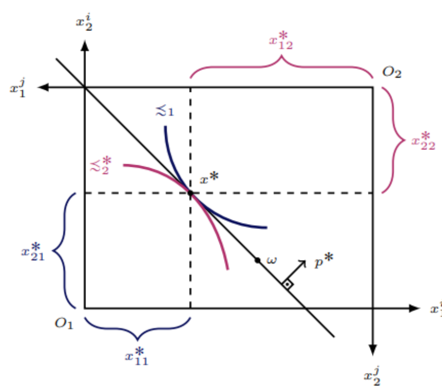

Figure 1: Edgeworth box.

Fore more details, see Microeconomic Theory by Mas-Colell et al. Chapter 15.

[1] Implicitly, unless the contrary is said, we assume that preferences are rational and continuous.

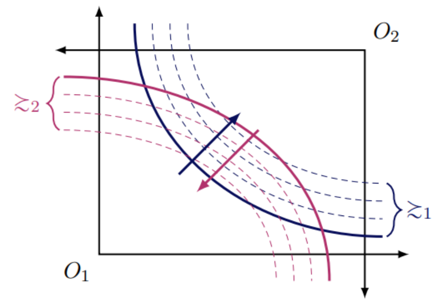

Figure 2: Preferences in $\square$.

The interest next is to study exchanges through the **market**. In this regard, we introduce a price vector $p = (p_1, p_2)$ associated with each consumer good. The prices set the rate of exchange for the goods: the good delivered must be worth the same or more than the good received. In this sense, if consumer i with initial endowment (ω_1^i, ω_2^i) wishes to consume (x_1^i, x_2^i), with $\omega_1^i \geq x_1^i$ and $\omega_2^i \leq x_2^i$, it must be satisfied that

$$p_1(\underbrace{\omega_1^i - x_1^i}_{\text{what is willing to give}}) \geq p_2(\underbrace{x_2^i - \omega_2^i}_{\text{what desires to obtain}}). \tag{1}$$

Rewriting 1 we get

$$\frac{p_2}{p_1} \leq \frac{\omega_1^i - x_1^i}{x_2^i - \omega_2^i}$$
$$p_1 x_1^i + p_2 x_2^i \leq p_1 \omega_1^i + p_2 \omega_2^i.$$

The first equation, when equality is achieved, tells us that individual i gives up $\omega_1^i - x_1^i$ units of good 1 in order to consume $\frac{p_2}{p_1}$ units of good 2. On the other hand, the second equation is known as the budget constraint of individual i. Given that $p \cdot \omega^i \in \{x^i : p \cdot x^i\}$ for $i = 1, 2$, the same line defines the budget constraint for each consumer in $\square$.

Proposition 1. The budget line is orthogonal to the price vector (p_1, p_2).

Proof. From the perspective of $i = 1$, an element on the budget line is of the form $a = \left(x_{a1}^1, \frac{I - p_1 x_{a1}^1}{p_2}\right)$. Another element would be $b = \left(x_{b1}^1, \frac{I - p_1 x_{b1}^1}{p_2}\right)$. Here

$I = p \cdot \omega^1$. Thus,

$$p \cdot (a - b) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \cdot \left[\begin{pmatrix} x^1_{a1} \\ \frac{I - p_1 x^1_{a1}}{p_2} \end{pmatrix} - \begin{pmatrix} x^1_{b1} \\ \frac{I - p_1 x^1_{b1}}{p_2} \end{pmatrix} \right] = 0.$$

$\square$

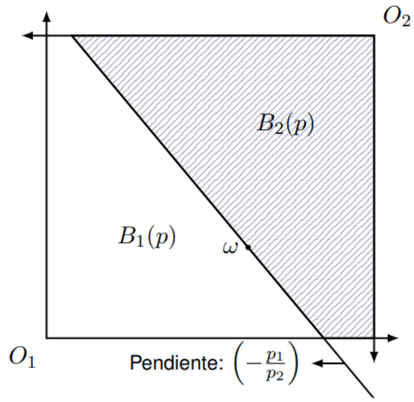

Figure 3: Budget set.

In the market, each consumer maximizes their utility subject to the budget constraint. That is, for $i = 1, 2$

$$\mathcal{P}_i : \begin{cases} \max & u^i(x^i_1, x^i_2) \\ \text{s.t.} & p_1 x^i_1 + p_2 x^i_2 \leq p_1 \omega^i_1 + p_2 \omega^i_2 \\ & x^i_1, x^i_2 \geq 0. \end{cases}$$

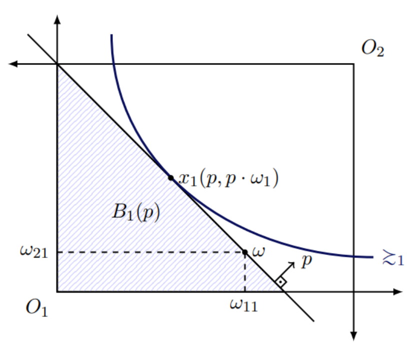

Figure 4: Optimal consumption.

Definition 2. We say that a market is in equilibrium if all consumers obtain what they wanted. These are the prices that balance the market: **equilibrium prices**. Formally, by solving each consumer's optimization problem $\mathcal{P}_i$, we obtain their demand correspondence

$$x^i(p) = (x_1^i(p_1, p_2), x_2^i(p_1, p_2))$$

and their **excess demand**

$$x^i(p) - \omega^i = (x_1^i(p_1, p_2), x_2^i(p_1, p_2)) - (\omega_1^i, \omega_2^i).$$

The condition that defines equilibrium prices is that the sum of excess demands from consumers equals zero. This is:

$$x^1(p) + x^2(p) - \omega^1 - \omega^2 = 0.$$

Note that these are two equations, which, broken down, are given by

$$x_1^1(p) + x_1^2(p) - \omega_1 = 0$$
$$x_2^1(p) + x_2^2(p) - \omega_2 = 0.$$

Proposition 3. Analytically, if solutions are interior and utility functions are differentiable and satisfy Inada's conditions[2], the consumer problem is solved by making

$$\frac{Umg_1^1}{Umg_2^1} = \frac{p_1}{p_2} = \frac{Umg_1^2}{Umg_2^2}$$

in conjunction with

$$p \cdot x^1(p) = p \cdot \omega^1(p)$$
$$p \cdot x^2(p) = p \cdot \omega^2(p)$$

and

$$x^1(p) - \omega^1 + x^2(p) - \omega^2 = 0.$$

Proof. The conditions

$$\frac{Umg_1^1}{Umg_2^1} = \frac{p_1}{p_2} = \frac{Umg_1^2}{Umg_2^2}$$

[2]So corner solutions are disregarded.

and

$$p \cdot x^1(p) = p \cdot \omega^1(p)$$
$$p \cdot x^2(p) = p \cdot \omega^2(p)$$

arise from the first order conditions of the Lagrangian function (for each i)

$$\mathscr{L}(x_1^i, x_2^i, \lambda) = U_i(x_1^i, x_2^i) + \lambda(p \cdot \omega^i - p \cdot x^i).$$

Finally, $x^1(p) - \omega^1 + x^2(p) - \omega^2 = 0$ is precisely the equilibrium price condition.

□

Definition 4. In a consumption scenario, if no improvement can be made for both consumers simultaneously—meaning neither consumer is strictly better off and at least one is strictly better than before—the allocation is referred to as a **Pareto Optimum** or **Pareto Efficient**.

We will return to this concept in much more detail later, but for now, we are making a very basic analysis in the case of 2x2 economies.

Proposition 5. The conditions for achieving a Pareto optimum (differentiable case with monotone preferences) are:

$$\frac{Umg_1^1(x^1)}{Umg_2^1(x^1)} = \frac{Umg_1^2(x^2)}{Umg_2^2(x^2)} \tag{2}$$
$$x^1 + x^2 = \omega^1 + \omega^2. \tag{3}$$

In particular, a **market equilibrium is a Pareto equilibrium**. Note that condition (1) is equivalent to $TMS_1 = TMS_2$.

Proof. The first condition (2) is a direct consequence of solving

$$\max U_i(x_1^i, x_2^i)$$
$$\text{s.t. } U_{-i}(x_1^{-i}, x_2^{-i}) \geq \overline{U}.$$

Indeed, the associated Lagrangian is

$$\mathscr{L}(x_1^1, x_2^1, x_1^2, x_2^2, \lambda) = U_i(x_1^i, x_2^i) + \lambda(\overline{U} - U_{-i}(x_1^{-i}, x_2^{-i})).$$

The second condition follows from the monotonicity of preferences. We leave the remaining details to the reader. □

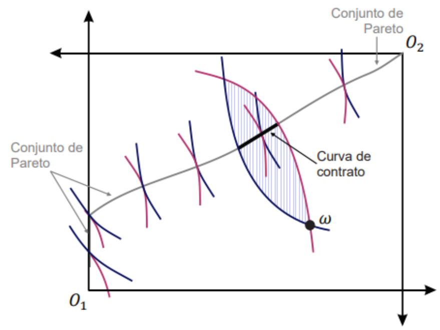

Figure 5: Pareto set.

Definition 6. The set of all Pareto allocations[3] is known as Pareto set.

We now preview the famous welfare theorems, but for the 2×2 case, and when preferences are represented by utility solutions. When we refer to transfers, we mean modifications to the budget constraints that involve adding $T_i \in \mathbb{R}$, where $T_i > 0$ represents a subsidy and $T_i < 0$ represents a tax. The statements below are informal. We will provide the formal statements when we address the case of pure exchange economies and with production.

Theorem 7. First Welfare Theorem. In the economy

$$\mathcal{E} = \{(u^1, \omega^1), (u^2, \omega^2)\}$$

if the utilities of the consumers are monotone, every Walrasian Equilibrium generates (is) a Pareto Optimum (allocation).

Theorem 8. Second Welfare Theorem. In the economy

$$\mathcal{E} = \{(U^1, \omega^1), (U^2, \omega^2)\}$$

if the utilities of the consumers are increasing, continuous, and strictly concave, every Pareto Optimum (allocation) corresponds to a Walrasian Equilibrium provided that an appropriate wealth transfer has been made beforehand.

[3] $\{(x_1^1, x_2^1), (x_2^1, x_2^2)\}$

1.1 Exercises

1. Suppose that in a 2×2 economy consumer i has Cobb-Douglas preferences $u_i(x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$. Furthermore, assume that endowments are $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$. Find Pareto optimal assignments and the (a) Walrasian equilibrium.

2. Consider a 2×2 economy such that

$$u_1(x_{11}, x_{21}) = x_{11} - \frac{1}{8} x_{21}^{-8}$$
$$u_2(x_{12}, x_{22}) = -\frac{1}{8} x_{12}^{-8} + x_{22}.$$

Consider the endowments, $\omega_1 = (2, r)$ and $\omega_(r, 2)$ with $r = 2^{8/9} - 2^{1/9} > 0$. Compute the offer curve[4] of each individual.

3. In each of the following cases, draw the Edgeworth box, some indifference curves for each consumer, the Pareto set and the core (contract curve). Finally, find Walrasian (competitive) equilibrium in each case.

a) $u_1(x_{11}, x_{21}) = 2x_{11}^2 x_{21}$, $u_2(x_{12}, x_{22}) = x_{12} x_{22}^3$, $\omega_1 = (2, 3)$ and $\omega_2 = (1, 2)$.

b) $u_1(x_{11}, x_{21}) = 2x_{11} + x_{21}$, $u_2(x_{12}, x_{22}) = x_{12} x_{22}^3$, $\omega_1 = (2, 3)$ and $\omega_2 = (1, 2)$.

c) $u_1(x_{11}, x_{21}) = x_{11} + \ln x_{21}$, $u_2(x_{12}, x_{22}) = x_{12} + 2 \ln x_{22}$, $\omega_1 = (2, 3)$ and $\omega_2 = (1, 2)$.

d) $u_1(x_{11}, x_{21}) = x_{11} x_{21}$, $u_2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\}$, $\omega_1 = (2, 6)$ and $\omega_2 = (4, 1)$.

e) $u_1(x_{11}, x_{21}) = \min\{2x_{11}, x_{21}\}$, $u_2(x_{12}, x_{22}) = \min\{x_{12}, 2x_{22}\}$, $\omega_1 = (1, 2)$ and $\omega_2 = (3, 4)$.

Identify whenever it is possible the «type» (Cobb-Douglas, CES, Leontief, linear...) of the utility function.

4. Consider a 2×2 economy in which consumers preferences are monotonic. Prove that

$$p_1 \left(\sum_{i=1}^{2} x_{1i}(p_1, p_2) - \omega_1 \right) + p_2 \left(\sum_{i=1}^{2} x_{2i}(p_1, p_2) - \omega_2 \right) = 0.$$

[4] Maximization points given the budget set, which depend on (p_1, p_2).

5. Consider and Edgeworth box economy in which each consumer has Cobb-Douglas preferences

$$u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$$
$$u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta},$$

with $\alpha, \beta \in (0,1)$. Consider endowments $(\omega_{1i}, \omega_{2i}) > 0$ for $i = 1, 2$. Solve for the equilibrium price ratio and allocation.

6. Show that if both consumers in an Edgeworth box economy have continuous, strongly monotone and strictly convex preferences, then the Pareto set has no holes, this is, it is a connected set.

7. Under some conditions over the preferences, in a 2×2 economy, every Pareto Optimal allocation can be characterized as the solution of the following maximization problem

$$\begin{aligned} &\max\ u_1(x_1) \\ &\text{s. t. } u_2(x_2) \geq k \\ &\qquad x_1 + x_2 = \omega_1 + \omega_2, \end{aligned}$$

where $k \in \mathbb{R}$. Find the conditions over the preferences.

8. There are two consumers, A and B, with the following utility functions,

$$u_A(x_A^1, x_A^2) = a \ln x_A^1 + (1-a) \ln x_A^2,\ \omega_1 = (0, 1)$$
$$u_B(x_B^1, x_B^2) = \min x_B^1, x_B^2,\ \omega_2 = (1, 0).$$

Calculate the prices and quantities that clear the market.

9. Consider two individuals in a pure exchange economy whose indirect utilities are

$$v_1(p_1, p_2, y) = \ln y - a \ln p_1 - (1-a) \ln p_2$$
$$v_2(p_1, p_2, y) = \ln y - b \ln p_1 - (1-b) \ln p_2$$

Endowments are $\omega_1 = (1, 1)$ and $\omega_2 = (1, 1)$. Obtain the prices that clean the market.

2 Robinson Crusoe Economy

In this economy, there is a single consumer, 2 goods, and there is production. Initially, the consumer possesses an initial quantity of only one of the goods and can produce the other good from it. The consumer certainly values both goods.

Formally, the consumer has an initial endowment: $\overline{L}$, their preferences are represented by a utility function $u(L, C)$, where L is for «leisure»[5] and c is the consumption of a certain good. Finally, the technology is determined by a function $f(\cdot)$ such that $C = f(L)$. For clarity, we will use the following notation:

1. L_o for the consumption of good L.

2. L_t for the use of good L in the production of C.

3. The consumer's problem is then to divide their initial endowment $\overline{L} = L_t + L_o$ in such a way that with $C = f(L_t)$ his utility is maximized. In other words, he solves

$$\begin{cases} \max & u(L_0, C) \\ \text{s.t.} & C = f(L_t) \\ & L_t + L_0 = \overline{L} \\ & L_t, L_o, C \geq 0. \end{cases} \tag{4}$$

For interior solutions, with differentiable utility and production functions

$$\frac{u_{mgL}(L_0, C)}{u_{mgC}(L_0, C)} = f'(L_t)$$

along with $C = f(L_t)$ and $L_t + L_o = \overline{L}$. This formulation (4) is known as the centralized formulation.

The solution to (4) represents the (unique) **Pareto efficient situation**. Now let's consider what happens if we introduce a **market**. Suppose both L and C are traded at the prices $P_L = w$ and $P_C = p$. On one side of the market, we have «Robinson consumer» offering labor L_t, consuming $L_o = \overline{L} - L_t$, and demanding C. On the other side, there is Crusoe Inc., demanding L_t to produce

[5]The consumers can use his time to work and produce C.

and supply C. The exchanges are made at market prices, resulting in profits for the company. Since consumer Robinson owns Crusoe Inc., these profits are incorporated into his budget constraint. In this scenario, which we will call the **market solution**, firs, the individual solves

$$\begin{cases} \max & \underbrace{pC - wL_t}_{\text{profit}} \\ \text{s. t.} & C = f(L_t) \\ & L_t, C \geq 0. \end{cases}$$

For interior solutions, when the production function is differentiable, the first-order condition provides

$$f'(L_t^d) = \frac{w}{p}.$$

Then, $C^* = f(L_t^d)$ and replacing in the profit function,

$$\Pi^* = pf(L_t^d) - wL_t^d.$$

In a second stage, we analyze the budget constraint of consumer Robinson Crusoe:

$$pC = wL_t + \Pi^*$$
$$pC + wL_o = w\overline{L} + \Pi^*.$$

Thus, he solves

$$\begin{cases} \max & u(L_o, C) \\ \text{s. t.} & pC + wL_o = w\overline{L} + \Pi^* \\ & C \geq 0 \\ & 0 \leq L_o \leq \overline{L}. \end{cases}$$

When the utility function is differentiable, for interior solutions the following first-order condition is obtained:

$$\frac{u_{mgL}(L_o^d, C^d)}{u_{mgC}(L_o^d, C^d)} = \frac{w}{p}$$

together with

$$pC^d + wL_o^d = w\overline{L} + \Pi^*.$$

Definition 9. Equilibrium occurs when the quantities demanded and supplied are equal:

$$L_t^d = L_t^s$$
$$C^d = C^s$$

and with $L_o = \overline{L} - L_t^s$.

Remark. In general, we will assume:

1. $f(0) = 0$, $f' > 0$, and $f'' < 0$.

2. $\frac{\partial u}{\partial C}, \frac{\partial u}{\partial L} > 0$ and $\frac{\partial^2 u}{\partial C^2}, \frac{\partial^2 u}{\partial L^2} < 0$ and $\frac{\partial^2 u}{\partial C^2}\frac{\partial^2 u}{\partial L^2} - \left(\frac{\partial^2 u}{\partial C \partial L}\right)^2 > 0$.

Market Equilibrium	Pareto Optimum
$\frac{umg_L(L_o,C)}{umg_C(L_o,C)} = \frac{w}{p} = f'(L_t)$	$\frac{umg_L(L_o,C)}{umg_C(L_o,C)} = f'(L_t)$
$C = f(L_t)$	$C = f(L_t)$
$L_t + L_o = \overline{L}$	$L_t + L_o = \overline{L}$

we see that:

1. The market equilibrium is a Pareto optimum (First Welfare Theorem).

2. The Pareto optimum is a market equilibrium (Second Welfare Theorem).

2.1 Exercises

1. For each of the following cases, draw in the space «hours-product» (L, C) the production function, hours limit, indifference curve, and budget set. Compute the optimal amount of labor hours and consumption of the other good (i) by solving the centralized problem, (ii) by means of a market structure. Compare and comment.

a) $u(\ell_o, c) = \ell_o^2 c$, $c = \sqrt{\ell_t}$ and $\overline{\ell} = 10$.

b) $u(\ell_o, c) = \ell_o^2 c$, $c = \ell_t$ and $\overline{L} = 10$.

c) $u(\ell_o, c) = \ell_o^2 c$, $c = \ell_t^2$ and $\overline{\ell} = 10$.

d) $u(\ell_o, c) = \ell_o + \sqrt{c}$, $c = \ell_t$ and $\overline{\ell} = 10$.

2. Prove that in an economy with one firm, one consumer, and strictly convex preferences and convex technology[6], the equilibrium level of production is unique.

3. Consider an economy with one firm and one consumer where $f(\ell_t) = \sqrt{\ell_t}$, $u(\ell_o, c) = \ln c + \ln \ell_o$ and $\overline{L} = 1$. Compute the equilibrium prices, profits, and consumptions.

[6] You can assume a concave production function.

3 The 2×2 production model

Consider an economy with J firms. Each firm j produces a consumer good q_j directly from a vector of L primary (i.e. nonproduced) inputs (factors) $z_j = (z_{1j}, ..., a_{Lj}) \geq 0$. Firm j's production takes place by means of a concave, strictly increasing an differentiable production function $f_j(z_j)$. The economy has total endowments of the L factor inputs $(\overline{z}_1, ..., \overline{z}_L) > 0$. These endowments are initially owned by consumers and have use only as production inputs. Let us suppose the prices of the J[7] produced consumption goods are fixed $p = (p_1, ..., p_J)$. This assumption relies on the fact that we are interested in the factor market(s). What we wish to determine if the equilibrium factor prices $w = (w_1, ..., w_L)$. Each firm solves

$$\max_{z_j \geq 0} \; p_j f_j(z_j) - w \cdot z_j.$$

The optimal j's firm demand is $z(p, w) \subset \mathbb{R}^L_+$. An equilibrium for the factor markets of this economy, given the fixed output prices p, is an input price vector $w^* = (w^*_1, \cdots, w^*_L)$ and a factor allocation

$$(z^*_1, \cdots, z^*_J) = ((z^*_{11}, \cdots, z^*_{L1}), \cdots, (z^*_{1J}, \cdots, z^*_{LJ}))$$

such that

$$z^*_j \in z_j(p, w), \; \forall \; j = 1, ..., J$$
$$\sum_j z^*_{j\ell} = \overline{z}_\ell, \; \forall \; \ell = 1, ..., L.$$

First order conditions provide

$$p_j \frac{\partial f_j(z^*_j)}{\partial z_{\ell j}} = w^*_\ell, \; \forall \; j = 1, ..., J, \; \ell = 1, ..., L$$
$$\sum_j z^*_{\ell j} = \overline{z}_\ell, \; \forall \; \ell = 1, ..., L.$$

The equilibrium output levels are then $q^*_j = f_j(z^*_j)$ for every j. Alternatively, FOC can be stated in terms of cost functions $c_j(w, q_j)$ for $j = 1, ..., J$. Output levels $(q^*_1, ..., q^*_J) > 0$ and factor prices w^* constitute (under this approach/for-

[7] Each firm produces a single good.

mulation) an equilibrium if and only if

$$p_j = \frac{\partial c_j(w^*, q_j^*)}{\partial q_j},\ j = 1, ..., J$$

$$\sum_{j=1}^{J} \frac{\partial c_j(w^*, q_j^*)}{\partial w_\ell} = \overline{z}_\ell,\ \forall\ \ell = 1, ..., L.$$

Note that the second condition can be re-stated by means of Shepard's Lema as follows:

$$\sum_{j=1}^{J} \frac{\partial c_j(w^*, q_j^*)}{\partial w_\ell} = \sum_{j=1}^{J} z_{\ell j}^* = \overline{z}_\ell.$$

Remark. From the central planner point of view, the welfare-maximizing property of competitive allocations lead to

$$\max_{(z_1,...,z_J)\geq 0} \sum_j (p_j f_j(z_j) - w^* \cdot z_j),\ \text{s. t.}: \sum_j z_j^* = \overline{z}.$$

However, since $\sum_j z_j^* = \overline{z}$, then $w^* \cdot z_j)$ is constant. Thus, the problem becomes

$$\max_{(z_1,...,z_J)\geq 0} \sum_j p f_j(z_j),\ \text{s. t.}: \sum_j z_j^* = \overline{z}.$$

Let us now be more specific and take $J = L = 2$. Hence, in this economy two outputs are produced by means of two inputs (factor). Let us assume that the production functions $f_1(z_{11}, z_{21}),\ f_2(z_{12}, z_{22})$ are homogeneous of degree one. Usually, factor 1 is labor and factor 2 capital.

For every vector of factor prices $w = (w_1, w_2)$, we denote $c_j(w)$ the minimum cost of producing one unit of good j and by $a_j(w) = (a_{1j}(w), a_{2j}(w))$ the input combination (assumed unique) at which the minimum cost is reached. By Shepard's Lema

$$\nabla c_j(w) = (a_{1j}(w), a_{2j}(w)).$$

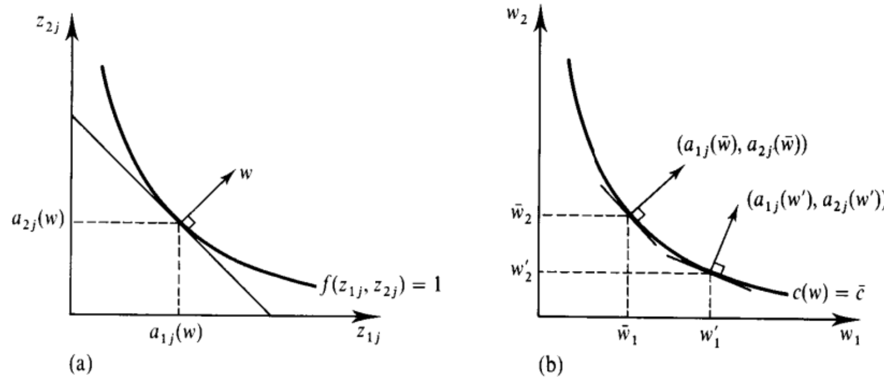

Figure 6: 2×2 model. From Mas-Colell et al. 1995 Chapter 15.

In Figure 6 (left side) it is observed the set

$$\{(z_{1j}, z_{2j}) \in \mathbb{R}^2_+ : f_j(z_{1j}, z_{2j}) = 1\},$$

along with the cost-minimizing input combination $(a_{1j}(w), a_{2j}(w))$. On the right side, we can observe a level set of the unit cost function

$$\{(w_1, w_2) : c_j(w_1, w_2) = \overline{c}\}.$$

This curve is downward slopping since as w_1 increases, w_2 must fall in order to keep the minimized cost of producing one unit of good j unchanged. Furthermore, the set

$$\{(w_1, w_2) : c_j(w_1, w_2) \geq \overline{c}\}$$

is convex because of the concavity of the cost function $c_j(w)$. Finally, note that as we move along the curve toward higher w_1 and lower w_2,

$$\frac{a_{1j}(w)}{a_{2j}(w)}$$

falls (why? what about the vector $\nabla c_j(\overline{w})$?).

1. Situate yourself in the Edgeworth box 2×2 for the factors z_1, z_2. Convince yourself that, when a Pareto Optimal allocation lies in the diagonal, then the Pareto set is precisely the diagonal. <u>Hint</u>: f_i is homogeneous of degree one, i.e., constant returns to scale.

Definition 10. The production of good 1 is relativelty more intensive in factor 1 than is the production of good 2 if

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

at all factor prices $w = (w_1, w_2)$.

Theorem 11. Stopler-Samuelson. In the 2×2 production model with the factor intensity assumption, if p_j increases, then the equilibrium price of the factor more intensively used in the production of good j increases while the price of the other actor decreases (assuming interior equilibrium both before and after the price change).

2. Prove Stopler-Samuelson theorem. For this, use the fact that

$$c_1(w_1, w_2) = p_1$$
$$c_2(w_1, w_2) = p_2.$$

Differentiating, you should obtain the system

$$dp = \underbrace{\begin{bmatrix} a_{11}(w^*) & a_{21}(w^*) \\ a_{12}(w^*) & a_{22}(w^*) \end{bmatrix}}_{=A} dw.$$

Factor intensity assumption implies that $|A| > 0$. Finally, take $dp = (1, 0)$ to conclude that $dw_1 > 0$ and $dw_2 < 0$.
Since $p_1 \uparrow$ implies $w_1^*/w_2^* \uparrow$, firms move to a less intensive use of factor 1. See Figure 7.

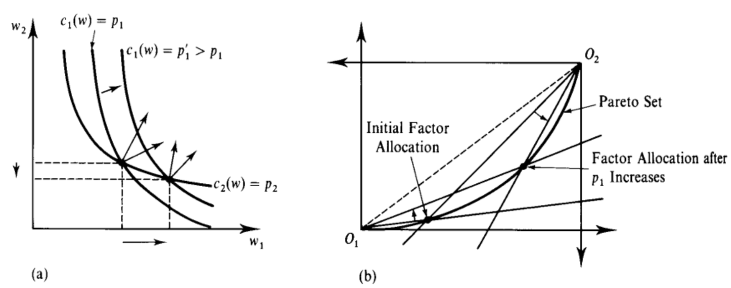

Figure 7: Stopler-Samuelson theorem.

Theorem 12. Rybcszynsky. In the 2×2 production model with the factor intensity assumption, if the endowment of a factor increases, then the production of the good that uses this factor relatively more intensively increases and the production of the other good decreases (assuming interior equilibria both before and after the change of endowment).

4 Pure Exchange Economy

We will start by addressing the basic pure exchange model:

1. A finite number of goods $\ell = 1, ..., L$ and consumers $i = 1, ..., I$.

2. Pure exchange (without production, firms).

3. Goods are bought and sold at prices $p \in \mathbb{R}^L_+$, which are uniform and fixed from the perspective of all consumers. Almost in every scenario, $p \in \mathbb{R}^L_{++}$. When $p_\ell = 0$ for a certain good, no one desires it.

4. Each consumer i has an endowment $\omega^i \in \mathbb{R}^L_+$ and their preferences $\succeq_i$ will be represented by a utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$. Moreover, $\overline{\omega} = \sum_{i=1}^I \omega^i >> 0$ is the economy (total) endowment.

5. The objective of the consumer is to choose the best element (maximal) according to their preferences within their budget set

$$B(p, \omega^i) = \left\{ x \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell x_\ell \leq \sum_{j=1}^{L} p_\ell \omega^i_\ell \right\}.$$

6. $\mathcal{E} = \{\succeq_i, \omega^i\}_{i=1,...,I}$ (when possible we replace $\succeq_i$ by u^i) is a pure exchange economy. Each consumer solves $\max_{x \in B(p,\omega^i)} u^i(x)$.

7. The set $B(p, \omega^i)$ is convex and compact for $p \in \mathbb{R}^L_{++}$. It is closed because $B(p, \omega^i) = f_p^{-1}(-\infty, p \cdot \omega^i]$ with $f_p(x) = p \cdot x$. It is bounded since $B(p, \omega^i) \subset \mathcal{B}_{||\cdot||_\infty}\left(0, \frac{2p \cdot \omega^i}{p_{\min}}\right)$. Hence, the budget set is compact, as claimed. Therefore, if the utility function (u^i) is continuous, the consumer's problem always has a solution (by the Weierstrass theorem). Moreover, if u^i is strictly quasi-concave, the solution is unique (see Mas-Colell et al., Chapter 3).

8. We define the demand function (correspondence to be formal, but we will mainly assume that it is a function) of each consumer i, as

$$\begin{aligned} x^i &: \mathbb{R}^L_{++} \to \mathbb{R}^L_+ \\ &p \to x^i(p) = \text{argmax}_{x \in B(p,\omega^i)} u(x). \end{aligned}$$

4.1 Efficiency

In general terms, we have the following situation: $i = 1, ..., I$ consumers who can consume $\ell = 1, ..., L$ goods. They "are born" with endowments ω^i and can trade, according to the exchange rates dictated by the price vector p, to obtain new consumption bundles x^i. A set of bundles $\{x^1, ..., x^I\}$ is known as an allocation, and we will now present a series of definitions and establish some key results to address the topic of efficiency in this context. For now, we assume there is no production.

Definition 13. Feasible Allocation. Given an economy $\mathcal{E} = \{(\omega^i, \succeq_i) : i = 1, ..., I\}$, a feasible allocation is a vector $(x^1, ..., x^I) \in \mathbb{R}_+^{L\times I}$ such that $\sum_{i=1}^I x^i \leq \sum_{i=1}^I \omega^i$.

Some times, we make the distinction between allocations $\leq$[8] and allocations $=$[9].

Definition 14. Pareto Optimality. An allocation $(x^i)_{i=1}^I$ is Pareto optimal in $\mathcal{E}$ if it is $\leq$ and there is no allocation $\leq$ $(y^i)_{i=1}^I$ such that $y^i \succeq_i x^i$ for every $i = 1, \ldots, I$, and $y^h \succ_h x^h$ for some $h \in \{1, \ldots, I\}$.

Sometime, the following definition is used.

Definition 15. Strong Pareto Optimality. A feasible allocation $(x^1, ..., x^I) \in \mathbb{R}_+^{L\times I}$ is a **strong Pareto optimum** if there does not exist $(\overline{x}^1, ..., \overline{x}^I) \in \mathbb{R}_+^{L\times I}$ such that

1. $u^i(\overline{x}^i) \geq u^i(x^i)$ for $i = 1, ..., I$.

2. $u^j(\overline{x}^j) > u^j(x^j)$ for some $j \in \{1, ..., I\}$.

Definition 16. Weak Pareto Optimality. A feasible allocation $(x^1, ..., x^I) \in \mathbb{R}_+^{L\times I}$ is a **weak Pareto optimum** if there does not exist $(\overline{x}^1, ..., \overline{x}^I) \in \mathbb{R}_+^{L\times I}$ such that $u^i(\overline{x}^i) > u^i(x^i)$ for $i = 1, ..., I$.

Definition 17. Walrasian Equilibrium. A Walrasian equilibrium in $\mathcal{E}$ is a pair (x, p): $x = (x^i)_{i=1}^I \in \mathbb{R}_+^{L\times I}$, and $p \in \mathbb{R}_+^L$ (a price vector), such that:

[8] $\sum_{i=1}^I x^i \leq \sum_{i=1}^I \omega^i$.
[9] $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$.

1. For every $i = 1, \ldots, I$, $x^i \in B(p, p\cdot\omega^i)$, and if $y^i \in B(p, p\cdot\omega^i)$ then $x^i \succeq_i y^i$ (all consumers optimize when choosing x^i at prices p);

2. $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$ (demand equals supply).

A characterization of Pareto optimal allocations. Let $\mathcal{E} = (\succeq_i, \omega^i)$ be an exchange economy in which each preference $\succeq_i$ is represented by a utility u_i. Consider the maximization problem $P(\overline{u})$:

$$(\text{PO}) : \begin{cases} \max_{x \in \mathbb{R}_+^{L \times I}} u_1(x_1) \\ \text{s.t. } u_i(x_i) \geq \overline{u}_i \ \forall i = 2, 3, \ldots, I, \\ \sum_{i=1}^I x_i \leq \overline{\omega}, \end{cases}$$

where $\overline{u} \in \mathbb{R}^I$ is a vector of utility values.

Proposition 18. Let each $\succeq_i$ be continuous and strictly monotone. Then, an allocation is a Pareto optimum iff there is $\overline{u} \in \mathbb{R}^I$ for which x solves $P(\overline{u})$.

Theorem 19. First Welfare Theorem. In a pure exchange economy, if preferences[10] $\succeq_i$ are locally non satiated, then every Walrasian equilibrium is Pareto efficient.[11]

Proof. See Chávez and Gallardo (2024) Chapter 10. You can prove it (it is not so hard). □

Definition 20. Walrasian Equilibrium with Transfers. A Walrasian equilibrium with transfers is a tuple (x, p, T), where $(x) \in \mathbb{R}_+^{L \times I}$, $p \in \mathbb{R}_+^L$ (a price vector), and $T = (T_i)_{i=1}^I \in \mathbb{R}^I$ (a vector of net transfers), such that:

1. For every $i = 1, \ldots, I$, $x^i \in B(p, M_i)$, and if $z^i \in B(p, M_i)$ then $x^i \succeq z^i$, where $M_i = p \cdot \omega^i + T_i$ (consumers optimize by choosing x^i in their budget sets);

2. $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$ (demand equals supply);

3. $\sum_{i=1}^I T_i = 0$ (net transfers are «budget balanced»).

[10] We always assume that these are rational. Actually, some texts such as Echenique's lecture notes, start defining preferences as a complete and transitive binary relation.

[11] The allocation of the Walrasian equilibrium is Pareto optimum.

Theorem 21. Second Welfare Theorem. In a pure exchange economy, if preferences are strongly monotone, convex and continuous[12] and x^* is a Pareto optimum such that $x^* >> 0$[13], then, there exists $p \in \mathbb{R}^L_{++}$ and $T \in \mathbb{R}^I$, $\sum_i T_i = 0$, such that (x^*, p^*, T) is a Walrasian equilibrium with transfers.

Proof. To prove this theorem, the separating hyperplane theorem is fundamental. You can find a proof in Chávez and Gallardo (2024) Chapter 10 or Federico Echenique's lecture notes. The proof is not as easy as the one for the First Welfare Theorem. $\square$

Definition 22. Coalition. A coalition is any non-empty subset of I.

Definition 23. Blocking. A coalition $S \subset I$ blocks a feasible allocation $(x^i)_{i=1,\dots,I} \in \mathbb{R}^{LI}_+$ if there exists an allocation $(\hat{x}^i)_{i\in S} \in \mathbb{R}^{L\times S}_+$ such that

1. For all $i \in S$: $\hat{x}^i \succ_i x^i$.
2. $\sum_{i\in S} \hat{x}^i \leq \sum_{i \in S} \omega^i$.

Definition 24. The core of an economy $\mathcal{N}(\mathcal{E})$ is the set of all feasible allocations that are not blocked by any coalition.

If $\mathcal{W}(\mathcal{E})$ is the set of all Walrasian equilibrium allocations, $\mathcal{P}(\mathcal{E})$ the set of Pareto optimum allocations, then when preferences are continuous, strictly convex, and strictly (strong) monotone $\mathcal{W}(\mathcal{E}) \subset \mathcal{N}(\mathcal{E}) \subset \mathcal{P}(\mathcal{E})$.

4.2 Excess of demand and existence of the Walrasian equilibrium

Definition 25. The excess demand function[14] of consumer i is

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i,$$

where $x_i(p, p\cdot\omega_i)$ is consumer's i Walrasian demand function. The (aggregate) excess demand function of the economy is

$$z(p) = \sum_i z_i(p).$$

[12]See Chávez and Gallardo (2024) for the definition.

[13]Note that it is implicitly required that $\overline{\omega} >> 0$, why?

[14]Formally, we should speak about correspondences. The generalization is not so hard but

Proposition 26. If $(\succeq_i, \omega_i)_{i=1}^I$ is an exchange economy in which $\overline{\omega} = \sum_{i=1}^I \omega_i > 0$ and each $\succeq_i$ is continuous, strictly convex and strictly monotone, then the aggregate excess demand function satisfies:

1. z is continuous.

2. z is homogeneous of degree zero.

3. Walras Law: $\forall\ p \in \mathbb{R}^L_{++}$: $p \cdot z(p) = 0$.

4. Bounded below: $\exists\ M > 0$ such that $\forall\ \ell, p \in \mathbb{R}^L_{++}$, $z_\ell(p) > -M$.

5. Boundary condition: if $\{p^n\}$ is a sequence in $\mathbb{R}^L_{++}$ and $\overline{p} = \lim_n p^n$ where $\overline{p} \in \mathbb{R}^L_0 \backslash \mathbb{R}^L_{++}$ and $\overline{p} \neq 0$, then there is $\ell \in \{1, ..., L\}$ such that $\{z_\ell(p^n)\}_n$ is unbounded.

Theorem 27. Let X be a non empty convex and compact subset of $\mathbb{R}^n$ and $f : X \to X$ a continuous function. Then, there exists $x^* \in X$ such that $f(x^*) = x^*$.

Theorem 28. Existence of Walrasian equilibrium. In the context of Proposition 26, for $z : \mathbb{R}^L_+ \to \mathbb{R}^L$, there exists $p^* \in \mathbb{R}^L_+$ such that $z(p^*) \leq 0$. Furthermore, if $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$, there exists p^* such that $z(p^*) = 0$.

Proof. First, since z is homogeneous of degree zero, we can restrict p to the Δ (also known as $n-$dimensional simplex), defined as follows:

$$\Delta = \left\{ p \in \mathbb{R}^L_+ : \sum_{\ell=1}^L p_\ell = 1 \right\}.$$

This set is clearly convex and compact. Indeed, given $p_1, p_2 \in \Delta$ and $\theta \in [0, 1]$,

$$p_3 = \theta p_1 + (1 - \theta) p_2 \in \Delta :$$

$$\begin{aligned} \sum_{\ell=1}^L p_\ell^3 &= \sum_{\ell=1}^L \theta p_\ell^1 + (1-\theta) p_\ell^2 \\ &= \theta \sum_{\ell=1}^L p_\ell^1 + (1-\theta) \sum_{\ell=1}^L p_\ell^2 \\ &= \theta + (1 - \theta) = 1. \end{aligned}$$

With respect to the compactness, Δ is closed since it is the intersection of the orthant $\mathbb{R}_+^L$ and the hyperplane $H\left((1,...,1),1\right)$. It is bounded since $\Delta \subset [0,1]^L$. Hence, since all of this occurs in $\mathbb{R}^L$, Δ is a compact set. It is therefore possible to apply Brouwer fixed point over Δ. We would only need to prove that $z(p)+p$ maps Δ onto Δ. However, this is not the case in general. This is where the following trick is employed, which allows us to conclude the matter using Brouwer's Fixed Point Theorem. Let us define $\Psi : \Delta \to \mathbb{R}^L$ defined as follows:

$$\Psi_\ell = \frac{p_\ell + \max\{0, z_\ell(p)\}}{1+\sum_{\ell=1}^L \max\{0, z_\ell(p)\}}, \ \forall\ \ell = 1,...,L.$$

Since $\sum_{\ell=1}^L p_\ell = 1$,

$$\sum_{\ell=1}^L \Psi_\ell = \sum_{\ell=1}^L \left\{ \frac{p_\ell + \max\{0, z_\ell(p)\}}{1+\sum_{\ell=1}^L \max\{0, z_\ell(p)\}} \right\} = 1,$$

i.e., $\Psi(\Delta) \subset \Delta$. Hence, by Theorem 27, there exists p^* such that $\Psi(p^*) = p^*$. This yields to: $\forall\ \ell = 1,...,L$

$$\begin{aligned}
p_\ell^* &= \frac{p_\ell^* + \max\{0, z_\ell(p^*)\}}{1+\sum_{\ell=1}^L \max\{0, z_\ell(p^*)\}} \\
p_\ell^*\left(1+\sum_{\ell=1}^L \max\{0, z_\ell(p^*)\}\right) &= p_\ell^* + \max\{0, z_\ell(p^*)\} \\
p_\ell^* \sum_{\ell=1}^L \max\{0, z_\ell(p^*)\} &= \max\{0, z_\ell(p^*)\} \\
z_\ell(p^*) p_\ell^* \sum_{\ell=1}^L \max\{0, z_\ell(p^*)\} &= z_\ell(p^*) \max\{0, z_\ell(p^*)\} \\
\underbrace{\sum_{\ell=1}^L z_\ell(p^*) p_\ell^*}_{=0} \left[\sum_{\ell=1}^L \max\{0, z_\ell(p^*)\}\right] &= \sum_{\ell=1}^L z_\ell(p^*) \max\{0, z_\ell(p^*)\}
\end{aligned}$$

Therefore,

$$\sum_{\ell=1}^L z_\ell(p^*) \max\{0, z_\ell(p^*)\} = 0. \tag{5}$$

for simplicity, we work only with functions, unless the contrary is said.

Equation 5 points out that $z_\ell(p^*) \leq 0,\ \forall\ \ell = 1, ..., L$. Finally, once again by Walras Law, since we must have

$$\sum_{\ell=1}^{L} p_\ell^* z_\ell(p^*) = 0 \tag{6}$$

with $p_\ell \geq 0$, combining (6) with Equation 5, we must have $p_\ell z_\ell(p^*) = 0$ for all $\ell = 1, ..., L$. Finally, for $p_\ell > 0$, necessarily $z_\ell(p^*) = 0$ for all $\ell = 1, ..., L$, which concludes the proof. Note that we are using the strict convexity of the preferences to ensure that z is a function. □

Some comments on the existence theorem of Walrasian equilibrium:

1. The argument is essentially topological as it makes use of Brouwer's theorem.

2. If we wanted to work with correspondences, it is imperative to use Kakutani's Theorem, which is a generalization of Brouwer's theorem.

3. For more details on the matter, see Echenique's lecture notes.

4. For our proof, which only uses Brouwer's fixed point theorem, we have followed Varian (Microeconomic Analysis) and Ellickson (Competitive Equilibrium Theory and Applications).

5. The theory of General Equilibrium (at least the one presented in these lecture notes, avoiding the differential approach or/and infinite goods) was developed by Kenneth Arrow, Gérard Debreu, and Lionel McKenzie.

4.3 Exercises

1. [Adapted from Aliprantis et al.] Consider an economy with 3 consumers and 2 goods. Utilities and endowments are given by

$$\begin{aligned}
u_1(x_{11}, x_{21}) &= x_{11}^{1/2} + x_{21}^{1/2},\ (\omega_{11}, \omega_{21}) = (1, 2) \\
u_2(x_{12}, x_{22}) &= \min\{x_{12}, x_{22}\},\ (\omega_{12}, \omega_{22}) = (3, 4) \\
u_3(x_{13}, x_{23}) &= x_{23} e^{x_{13}},\ (\omega_{13}, \omega_{23}) = (1, 1).
\end{aligned}$$

Prove that the optimal demands are given by

$$x_{11} = \frac{p_2 p_1 + 2p_2^2}{p_1^2 + p_2 p_1}, \ x_{21} = \frac{p_1^2 + 2p_2 p_1}{p_2 p_1 + p_2^2}$$
$$x_{12} = x_{22} = \frac{3p_1 + 4p_2}{p_1 + p + 2}$$
$$x_{13} = \frac{p_2}{p_1}, \ x_{23} = \frac{p_1}{p_2}.$$

2. Find the optimal demands in a pure exchange economy with L consumption goods, N consumers, where each consumer $k = 1, ..., N$ has preferences represented by

$$u_k(x_k) = \prod_{\ell=1}^{L} x_{\ell k}^{\alpha_{\ell k}},$$

$\sum_{\ell=1}^{L} \alpha_{\ell k} = 1, \ \alpha_{\ell k} \in (0,1)$, and endowments $\omega_k > 0$. Do not seek to find the Walrasian equilibrium.

3. Consider an economy with N consumers, two goods, and preferences given by

$$u_i(x_{1i}, x_{2i}) = x_{1i}^2 + x_{2i}^2.$$

Endowments are $\omega_i = (1,1)$. If N is even, find, if it exists, a Walrasian equilibrium. What if N is odd?

4. Consider a pure exchange economy where all consumers have the same preferences. Under what (minimal) conditions over the preferences, an allocation where every single individual consumes the same bundle is Pareto efficient?

5. Prove Proposition 26. You will need Berge theorem.

6. Let $z(p_1, p_2) = \left(\frac{Bp_2}{p_1}, \frac{Ap_1}{p_2}\right) - (A, B)$. Prove that z satisfies the five properties of an excess demand function.

7. Consider a 2×2 economy where the first consumer has preferences represented by a Cobb-Douglas utility function

$$u_1(x_{11}, x_{21}) = x_{11} x_{21}$$

and initial endowment $\omega_1 = (2, 6)$. The second consumer has preferences

$$u_2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\}$$

and initial endowment $\omega_2 = (4, 1)$. Let $p = (p_1, p_2) \in \mathbb{R}_{++}^2$.

a) Find the demand (correspondence) of each consumer.

b) Find the excess demand (correspondence) of each consumer.

c) Verify if z satisfies the usual properties of excess demand functions.

d) ¿Is there an equilibrium in this economy?

e) Find the Pareto optimal allocations.

8. There is an alternative approach to characterizing Pareto efficient allocations that is sometimes useful. In this approach, one considers maximizing a linear (Bergson-Samuelson) social welfare function of the form $\sum_i \alpha_i u_i$ subject to a resource constraint. The program is:

$$\max_{x_1,\ldots,x_I} \sum_i \alpha_i u_i(x_{i1},\ldots,x_{iL})$$

subject to

$$\sum_i x_i \leq \sum_i \omega_i.$$

Prove this equivalence. Hint: apply FOC to the PO problem.

5 Economies with production

1. Economic agents are capable of transforming bundles of goods.

2. Each firm $j = 1, ..., J$ is defined by its possibilities of transforming bundles, which we will call technology. These are denoted Y_j.

3. They maximize their profit.

4. y^D is the bundle of goods used as input and y^O is the bundle of goods produced by the firm.

5. If y is a production plan, $y^D = -\min\{y, 0\}$. On the other hand, the supply is $y^O = \max\{y, 0\}$.

$$y = y^O - y^D = \max\{y, 0\} + \min\{y, 0\}.$$

6. If $y_\ell < 0$, good ℓ is used as input, in the amount $|y_\ell| = -y_\ell$.

Note. From now, we will assume that technologies are non empty closed sets.

Remark. Sometimes we can write Y by means of a function $F : \mathbb{R}^L \to \mathbb{R}$

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}.$$

When there is a single product,

$$Y = \{(y, -x) \in \mathbb{R}^L : y \leq f(x),\ y, x \geq 0\}.$$

Definition 29. A technology $Y \subset \mathbb{R}^L$ exhibits

1. Possibility of inactivity: $0 \in Y$.

2. Possibility of free disposal: if $y \in Y$ and $y' \leq y$ (in each component), then $y' \in Y$.

3. No free lunch: if $y \in Y$ with $y \geq 0$, then $y = 0$.

4. Strictly bounded: if there exists $K \in \mathbb{R}$ such that every $y \in Y$ satisfies $y_\ell \leq K$.

Definition 30. A technology $Y \subset \mathbb{R}^L$ is convex if it is a convex set. Moreover, it is strictly convex if $\forall\ y_1, y_2 \in Y$ and $\alpha \in (0,1)$:

$$\alpha y_1 + (1-\alpha) y_2 \in Y^\circ.$$

Given the prices $p >> 0$, the profit from the plan y is

$$\begin{aligned} py &= p \max\{y, 0\} + p \min\{y, 0\} \\ &= p \max\{y, 0\} - p(-\min\{y, 0\}) \\ &= py^O - py^D \\ &= I_y - C_y \end{aligned}$$

where I_y is the revenue from sales and C_y is the cost. The objective of the firm is to solve

$$\begin{aligned} &\max\ py \\ &\text{s.t. } y \in Y. \end{aligned}$$

Theorem 31. If the set Y is compact and strictly convex, then the problem has a solution, which is unique $\hat{y} \in Y$ and such that $p\hat{y} \geq py,\ \forall\ y \in Y$.

The theorem ensures the well-defined nature of $\pi(p) = \max_{y \in Y}\{py\}$ and of the supply function $y(p) = \text{Argmax } py$ s.t. $y \in Y$. Note that $\pi(p) = py(p)$.

Theorem 32. Given a technology Y that is non-empty, strictly convex, closed, and bounded above, the supply satisfies:

1. It is continuous.
2. It is homogeneous of degree zero.
3. It is bounded above.

On the other hand, the profit function is

1. Continuous.
2. Homogeneous of degree one.
3. Convex.

Proof. The proof requires Theorem **??** for all the results related to continuity. The other properties are straightforward to derive. □

5.1 Private ownership economies

There are L goods, I consumers, and J firms:

1. Each firm j has a technology Y_j.
2. Each consumer i has an initial endowment ω_i, a preference $\succeq_i$ over the consumption space, and shares in the profits of each firm, stacked in the vector θ_i: $\theta_i = (\theta_{i1}, ..., \theta_{iJ})$, $0 \le \theta_{ij} \le 1$ and $\sum_{i=1}^{I} \theta_{ij} = 1$.

Then, a private ownership economy is the collection (or tuple)

$$\mathcal{E} = \{(\omega_i, \succeq_i)_{i=1,\dots,I}, (Y_j)_{j=1,\dots,J}, (\theta_{ij})_{i=1,\dots,I,j=1,\dots,J}\}.$$

We assume that, for $\overline{\omega} = \sum_{i=1}^{I} \omega_i$ and $Y = \sum_{j=1}^{J} Y^j$, there exists $y' \in Y$ such that

$$\overline{\omega} + y' >> 0.$$

In other words, there exists a production plan such that, starting from the initial endowment, we can obtain positive quantities of all goods. Viewed another way, if there are no units of a good, it can be produced in some quantity without depleting another of the economy's initial goods.
In an economy with production $\mathcal{E}$, each firm j, by solving its profit maximization problem

$$\max\ py$$
$$\text{s.t. } y \in Y^j$$

generates a supply $y^j(p)$ and profits $\pi^j(p) = py^j(p)$. In turn, each consumer i solves

$$\begin{aligned} &\max\ u^i(x) \\ &\text{s.t. } px \le p\omega_i + \sum_{j=1}^{J} \theta_{ij} p y^j(p) \\ &\qquad x \ge 0 \end{aligned}$$

and generates a demand $x^i(p)$.

Note. From now on, we assume that

1. $\succeq_i$ are rational, continuous, strictly convex, and strongly monotonic.

2. Y^j are closed, bounded above, strictly convex, and with $0 \in Y^j$.

Theorem 33. For every $p \in \mathbb{R}^L_{++}$, there exists a unique $x^i(p) \in \mathbb{R}^L_+$ solution to the problem of the economy $\mathcal{E}$, which satisfies:

1. $\forall\ \alpha > 0:\ x^i(\alpha p) = x^i(p)$ - homogeneous of degree zero.

2. $px^i(p) = p\omega^i + \sum_{j=1}^J \theta_{ij} p y_j(p)$ - Walras' law.

3. $x^i : \mathbb{R}^L_{++} \to \mathbb{R}^L_+$ is continuous.

Theorem 34. For any sequence (of vector prices) that converges to a point on the boundary of $\mathbb{R}^L_{++}$, that is, $p^n \to p$, where p is non-zero but there exists an ℓ such that $p_\ell = 0$, either

$$\max_\ell\ x^i_\ell(p^n) \to \infty, \quad \text{for some } i = 1, ..., I$$

or

$$\min_\ell\ y^j_\ell(p^n) \to -\infty, \quad \text{for some } j = 1, ..., J.$$

Definition 35. In a production economy, the excess demand function is

$$z(p) = \sum_{i=1}^I x_i(p) - \sum_{j=1}^J y_j(p) - \sum_{i=1}^I \omega^i.$$

Definition 36. For an economy $\mathcal{E}$ with an aggregate excess demand function $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$, we say that p^* is an equilibrium price if $z(p^*) = 0$.

Theorem 37. In the case of an economy with production, the properties of the excess demand function from a pure exchange economy still hold.

Definition 38. Given an economy with production

$$\mathcal{E} = \{(\omega_i, \succeq_i)_{i=1,...,I}, (Y_j)_{j=1,...,J}, (\theta_{ij})_{i=1,...,I, j=1,...,J}\}$$

a feasible allocation is a vector $(x^1, ..., x^I, y^1, ..., y^J) \in \mathbb{R}^{IL}_+ \times \mathbb{R}^{JL}$ such that

$$\sum_{i=1}^I x^i \leq \sum_{i=1}^I \omega^i - \sum_{j=1}^J y^j.$$

Definition 39. An allocation $(x, y) \in \mathbb{R}_+^{IL} \times \mathbb{R}^{JL}$ is an equilibrium if there exists $p \in \mathbb{R}_{++}^L$ such that:

1. For all $j = 1, ..., J$: $py^j \geq py'$ for all $y' \in Y^j$: firms maximize.
2. For all $i = 1, ..., I$: $px^i \leq p\omega^i + \sum_{j=1}^J \theta_{ij} py^j$: feasibility of the allocations.
3. For all $i = 1, ..., I$: $u^i(x) > u^i(x^i)$ implies $px > p\omega^i + \sum_{j=1}^J \theta_{ij} py^j$: consumers maximize.

Definition 40. Pareto Optimality. An allocation (x, y) in $\mathcal{E}$ is Pareto optimal[15] if there is no allocation $(\tilde{x}, \tilde{y})$ such that $\tilde{x}_i \succeq_i x_i$ for every $i = 1, \ldots, I$, and $\tilde{x}_h \succ_h x_h$ for some $h \in \{1, \ldots, I\}$.

Definition 41. Weak Pareto Optimality. An allocation (x, y) is a weak Pareto optimum if

1. It is feasible.
2. $\nexists(\tilde{x}, \tilde{y})$ feasible such that $\forall\ i = 1, ..., I$, $\tilde{x}^i \succeq x^i$ and for at least one $h \in \{1, ..., I\}$, $\tilde{x}^h \succ x^h$.

Theorem 42. First Welfare Theorem in a POE. Assume that in a POE consumers' preferences are locally nonsatiated. Then, every equilibrium allocation is a strong Pareto optimum.

Definition 43. Walrasian Equilibrium. Let $\mathcal{E}$ be a private ownership economy. A Walrasian Equilibrium is a pair $(x, y) \in \mathbb{R}_+^{IL} \times \mathbb{R}_+^{JL}$, together with a price vector $p \in \mathbb{R}_+^L$ such that:

1. For every $i = 1, ..., I$, $x_i \in B(p, M_i)$, and $x_i' \in B(p, M_i) \Rightarrow x_i \succeq x_i'$, where $M_i = p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y_j$ (consumers optimize by choosing x_i in their budget sets);
2. For every $j = 1, ..., J$, $y_j \in Y_j$, and $p \cdot y_j \geq p \cdot y_j'\ \forall\ y_j' \in Y_j$ (firms optimize profits by choosing y_j in Y_j);
3. $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j$ (demand equals supply).

[15] Also referred as strong Pareto allocation or Pareto optimum.

There is a simple and useful characterization of Pareto optimal allocations. Let $((Y_j)_{j=1}^J, (\succeq_i, \omega_i, \theta_i)_{i=1}^I)$ be a private ownership economy in which each preference relation $\succeq_i$ has a continuous and strictly monotone utility representation $u_i : \mathbb{R}_+^L \to \mathbb{R}$. Consider the following optimization problem:

$$\max_{(x,y)\in\mathbb{R}_+^{IL}\times\mathbb{R}_+^{JL}} u_1(x_1) \quad \text{(PO)}$$

subject to

$$u_i(x_i) \geq \overline{u}_i \ \forall\ i = 2, 3, \ldots, I,$$
$$\sum_{i=1}^I x_i \leq \overline{\omega} + \sum_{j=1}^J y_j,$$
$$y_j \in Y_j \ \forall j = 1, \ldots, J.$$

Proposition 44. Suppose that each preference $\succeq_i$ is continuous and strictly monotone. An assignment (x, y) solves the maximization problem (PO) if and only if it is Pareto Optimal.

Definition 45. Walrasian Equilibrium with Transfers. A Walrasian equilibrium with transfers is a tuple (x, y, p, T), where $(x, y) \in \mathbb{R}_+^{IL} \times \mathbb{R}_+^{JL}$, $p \in \mathbb{R}_+^L$ (a price vector), and $T = (T_i)_{i=1}^I \in \mathbb{R}^I$ (a vector of net transfers), such that:

1. For every $i = 1, \ldots, I$, $x_i \in B(p, M_i)$, and $x_i' \in B(p, M_i) \Rightarrow x_i \succeq x_i'$, where $M_i = p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y_j + T_i$ (consumers optimize by choosing x_i in their budget sets);

2. For every $j = 1, \ldots, J$, $y_j \in Y_j$, and $p \cdot y_j \geq p \cdot y_j' \ \forall\ y_j' \in Y_j$ (firms optimize profits by choosing y_j in Y_j);

3. $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j$ (demand equals supply);

4. $\sum_{i=1}^I T_i = 0$ (net transfers are "budget balanced").

Theorem 46. Second Welfare Theorem in a POE. Let

$$\mathcal{E} = (\succeq_i, \omega_i, \theta_i)_{i=1}^I, (Y_j)_{j=1}^J$$

be a P.O.E. in which each Y_j is closed and convex, and each preference $\succeq_i$ is strongly monotone, convex, and continuous. If (x^*, y^*) is a Pareto optimal allocation in which $\sum_{i=1}^{I} x_i^* \geq 0$, then there is a price vector $p^* \in \mathbb{R}_+^L$ and transfers $T = (T_i)_{i=1}^{I}$ such that (x^*, y^*, p^*, T) is a Walrasian equilibrium with transfers.

The proof of the welfare theorems in the context of a P.O.E. is analogous to that of pure exchange economies. You are encouraged to prove them.

5.2 Exercises

1. Prove Theorems 33 and 34. You are encouraged to interpret them as well.

2. Study the properties of the technology

$$Y = \left\{(x, y) \in \mathbb{R}^2 : x < 1, y \leq \frac{x}{x-1}\right\}.$$

In particular: closeness, convexity and free disposal.

3. Consider a firm with technology $Y = \{(-x, z) \in \mathbb{R}^2 : x \geq 0, z \leq f(x)\}$. Prove that if Y possess the free disposal property, then f is non decreasing.

4. Consider an economy with two goods, two consumers and one firm. Consumer 1 has preferences represented by

$$u_1(x_{11}, x_{21}) = \sqrt{x_{11}x_{21}},$$

with initial endowment $\omega_1 = (1, 0)$ and $\theta_1 = 0.3$. Consumer 2 has quasilinear preferences

$$u_2(x_{12}, x_{22}) = x_{12} + \ln(x_{22}),$$

with initial endowment $\omega_2 = (2, 0)$ and $\theta_2 = 0.7$. On the other hand, the firms technology is

$$Y = \left\{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq \frac{Ax}{x-1}\right\}$$

where $A > 0$ is a productivity factor.

1. Find the offer function of the firm.

2. Find each consumers correspondence demand.

3. Find the excess demand correspondence of the economy $z(p_1, p_2)$.

4. Study the effect of the productivity factor A over the equilibrium (prices and allocation). In other terms, do some comparative statics focusing on the parameter A.

5. Consider and economy with two consumers:

$$u_A(x_{A1}, x_{A2}) = \min\left\{x_{A1}, \frac{x_{A2}}{4}\right\},\ \omega_A = (a, 1), \theta_A = 1/3$$
$$u_B(x_{B1}, x_{B2}) = (x_{B1})^{1/3}(x_{B2})^{2/3},\ \omega_B = (1, b),\ \theta_B = 2/3,$$

with $a, b > 0$. Let

$$Y = \{(x_1, x_2) : 4x_2 + x_1 \leq 0,\ 4x_1 + x_2 \leq 0\}$$

be the firms technology.

a) Set the firm problem and solve it; specify all the price vector $p \in \Lambda$ for which the problem has a solution. Obtain the offer and profit correspondences.

b) Consider a specific $p \in \Lambda$. Set and solve the consumers problem (for each one).

c) Obtain the excess demand function and analyze if it satisfies the basic properties[16].

6. Consider an economy with two goods, two consumers and a firm. Consumers have quasilinear utilities:

$$u_1(m_1, x_1) = m_1 + 4\ln x_1$$
$$u_2(m_2, x_2) = m_2 + \ln x_2.$$

Initial endowments are $\omega_1 = (100, 0)$ and $\omega_2 = (100, 0)$. Each one owns a fraction θ_i of a firm whose technology is given by

$$Y = \{(-m_e, x_e) : x_e = \sqrt{m_e},\ x_e \geq 0, m_e \geq 0\}.$$

We take $x_i \geq 0$ but $m_i \in \mathbb{R}$. This is, consumers can consume a negative amount of m. Let p_m be the price of good m and p_x the price of good x.

[16] They are analogous to the Pure Exchange Economies case. See Mas-Colell et all (1995)

1. Find the firm's offer.
2. Find each consumers demand.
3. Find the aggregated excess demand function.
4. There is a property which is not satisfied[17], which one? Why?
5. Can you normalize $p_m = 1$? Justify.
6. Prove that, in this economy, the equilibrium prices do not depend on the initial wealth[18] distribution.

for a more detailed discussion.

[17]From the properties that aggregated demand function satisfy.

[18]Endowments and shares.

6 Uniqueness of the Walrasian equilibrium

The issue of the existence of Walrasian equilibrium has already been discussed previously. In this section, we will discuss the question of uniqueness.

Example 47. Consider 2 individuals with preferences

$$u^1(x_1, x_2) = x_1 - \frac{1}{8}x_2^{-8}$$
$$u^2(x_1, x_2) = x_2 - \frac{1}{8}x_1^{-8}$$

with endowments $\omega^1 = (2, r)$ and $\omega^2 = (r, 2)$, $r > 0$. These utilities represent rational, continuous, convex, and strictly monotonic preferences. Normalizing $p_2 = 1$,

$$z(p_1) = \frac{r}{p_1} - \frac{1}{p_1^{8/9}} + \frac{1}{p_1^{1/9}} - r.$$

We quickly note that $z_1(1, 1) = 0$. However, for $r = 2^{8/9} - 2^{1/9}$, we find another equilibrium. We also note that if $(p^*, 1)$ is an equilibrium, $(1/p^*, 1)$ is also an equilibrium. Thus, we do not have guaranteed uniqueness.

Theorem 48. Sonnenschein-Mantel-Debreu. Given Z that satisfies Proposition 26, there exists an economy $\mathcal{E}$ that generates it as its aggregate demand function.

The proof of Sonnenschein-Mantel-Debreu is not trivial at all.

Remark. While more restrictions must be imposed on the economy to ensure the uniqueness of an equilibrium, every economy has a finite and odd number of isolated equilibria.

Let us normalize $p_L = 1$ and define

$$\hat{z}(p) = (z_1(p), z_2(p), \ldots, z_{L-1}(p)).$$

Also, let $\hat{D}(p) = D(p_1, \ldots, p_{L-1})\hat{z}(p)$.

Definition 49. An equilibrium p^* is regular if the matrix $\hat{D}(p^*)$ is regular, i.e., it has a non-zero determinant. An economy is regular if all its equilibriums are regular. An economy that is not regular is called critical.

Remark. Regular economies are dense.

Definition 50. Local Uniqueness. An equilibrium $p^* \in \mathcal{P}(\vec{\omega})$ is locally unique if there exists $\epsilon > 0$ such that, $\forall p \in \mathbb{R}^{L-1}_{++}$,

$$\|p - p^*\| < \epsilon \implies p \notin \mathcal{P}(\vec{\omega}),$$

where $\mathcal{P}(\vec{\omega})$ is the set of equilibrium prices.

Proposition 51. Let $\hat{z}$ be C^1 and $p^* \in \mathcal{P}(\vec{\omega})$ be a regular equilibrium. Then, p^* is locally unique. Furthermore, there are neighborhoods B_1 of $\vec{\omega}$ in $\mathcal{E}$, and B_2 of p^* in $\mathbb{R}^{L-1}_{++}$, and a function $h : B_1 \to B_2$ such that

$$\hat{z}(h(\vec{\omega}), \vec{\omega}) = 0 \quad \forall\, \vec{\omega} \in B_1,$$

and

$$D_{\vec{\omega}} h(\vec{\omega}) = -\left[D_p \hat{z}(p, \vec{\omega}) \Big|_{p=p^*} \right]^{-1} \cdot D_{\vec{\omega}} \hat{z}(p^*, \vec{\omega}).$$

Definition 52. Index. The index of $p \in \mathcal{P}(\vec{\omega})$ is defined as:

$$\text{index}(p) = (-1)^{L-1} \cdot \text{sign}\left(\det\left(D_p \hat{z}(p, \vec{\omega})\right)\right),$$

where $\det\left(D_p \hat{z}(p, \vec{\omega})\right)$ is the determinant of the matrix $D_p \hat{z}(p, \vec{\omega})$.

Note that for every regular economy $\vec{\omega}$, $\text{index}(p) \in \{-1, 1\}\ \forall\, p \in \mathcal{P}(\vec{\omega})$.

We state without proof the following theorem.

Theorem 53. Index Theorem. If $\vec{\omega}$ is a regular economy, then

$$\sum_{p \in \mathcal{P}(\vec{\omega})} \text{index}(p) = 1.$$

The index theorem can be used to establish uniqueness: if you can show that any competitive equilibrium in an economy has index one, then there can only be one equilibrium. Finally, the index theorem implies the following curiosity.

Corolario 54. A regular economy has an odd number of equilibria.

Definition 55. Given an excess demand function (EDF) z, we say that it satisfies the Weak Axiom of Revealed Preferences (WARP) if for any pair of prices p, p' such that $z(p) \neq z(p')$, if $pz(p') \leq 0$, then $p'z(p) > 0$.

Definition 56. An EDF z satisfies the property of Gross Substitutes if for any pair of prices p and p' such that $p' = p + \varepsilon \mathbf{e}_\ell$ ($\varepsilon > 0$),

$$z_k(p) > z_k(p'),\ \ k \neq \ell.$$

Theorem 57. If, in a regular economy, an EDF z satisfies the Gross Substitutes property and conditions from Proposition 26, it has a unique equilibrium.

7 The core

Let $\mathcal{E} = \{\omega^i, \preceq_i \colon\ i = 1, ..., I\}$ be a pure exchange economy. Following Federico Echenique's notation:

- An allocation $\leq$ in E is a vector $x = (x_i)_{i=1}^I \in \mathbb{R}_+^{IL}$, such that $\sum_{i=1}^I x_i \leq \sum_{i=1}^I \omega_i = \overline{\omega}$.
- An allocation $=$ in E is a vector $x = (x_i)_{i=1}^I \in \mathbb{R}_+^{IL}$, such that $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i = \overline{\omega}$.
- A nonempty subset $S \subseteq \{1, \ldots, I\}$ of agents is called a coalition.
- Let S be a coalition. A vector $(y_i)_{i \in S}$ is an S-allocation $\leq$ if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$.
- Let S be a coalition. A vector $(y_i)_{i \in S}$ is an S-allocation $=$ if $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$.

Definition 58. We say that

- A coalition S *blocks* the allocation $\leq$ x in E if there exists an S-allocation$\leq$ $(y_i)_{i \in S}$ such that $y_i > x_i$ for all $i \in S$.
- An allocation$\leq$ is *weakly Pareto optimal* if it is not blocked by the coalition I of all consumers.
- It is *individually rational* if no coalition consisting of a single consumer blocks it.
- It is a *core allocation* if there is no coalition that blocks it.

Let $C(\mathcal{E})$ be the set of core allocations$\leq$ of $\mathcal{E}$. We refer to $C(\mathcal{E})$ as the core of the economy $\mathcal{E}$. Let $\mathcal{P}(\mathcal{E})$ be the set of Pareto Optimal allocations$\leq$ of the economy $\mathcal{E}$, and let $\mathcal{W}(\mathcal{E})$ be the set of Walrasian Equilibrium allocations$\leq$. Note that $C(\mathcal{E})$, $\mathcal{W}(\mathcal{E})$, and $\mathcal{P}(\mathcal{E})$ are subsets of $\mathbb{R}_+^{IL}$.

Definition 59. A coalition S *weakly blocks* the allocation $\mathbf{x}$ if there exists an S-allocation$\leq$ $(y_i)_{i \in S}$ such that $y_i \geq x_i$ for all $i \in S$, and $y_j > x_j$ for some $j \in S$.

1. Prove that, if each $\succeq_i$ is continuous and strictly monotonic, then a coalition blocks an allocation if and only if it weakly blocks it. Hint: the surplus can be divided. Consider $z_i = (1-\delta)y_i$ por δ small enough, and $z_j = \frac{\delta y_i}{|S|-1} + y_i$.

Note. Form now, preferences will be continuous and strictly monotonic.

2. Conclude that, from Note 7, the only relevant allocations will be $=$.

Remark. Note that:

- If each preference relation is continuous and strictly monotonic, then all core allocations are Pareto Optimal, i.e., $C(E) \subseteq \mathcal{P}(E)$.
- An allocation $\mathbf{x}$ of E is individually rational if $x_i \succeq_i \omega_i$ for all $i = 1, \ldots, I$.
- If $x \in C(E)$, then x is individually rational.

Theorem 60. Every Walrasian Equilibrium allocation is a core allocation, i.e., $\mathcal{W}(\mathcal{E}) \subset C(\mathcal{E})$.

3. Prove Theorem 60.

Definition 61. Replica Economy. Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy and $N \geq 1$ be an integer. The N-th replica of $\mathcal{E}$, denoted as $\mathcal{E}_N = (\succeq_{i,n}, \omega_{i,n})_{i=1,\ldots,I,n=1,\ldots,N}$, is an exchange economy where agents are indexed by (i,n), and it holds that for every $n = 1, \ldots, N$, $\succeq_i = \succeq_{i,n}$ and $\omega_i = \omega_{i,n}$. Note that the replica $\mathcal{E}_N$ comprises IN agents.

Definition 62. Equal Treatment Property. An allocation

$$x = (x_{i,n})_{i=1,\ldots,I,n=1,\ldots,N}$$

of $\mathcal{E}_N$ exhibits the equal treatment property if $x_{i,n} = x_{i,m}$ for all $n, m = 1, \ldots, N$ and $i = 1, \ldots, I$.

Lemma 63. Assume $\succeq_i$ is strictly monotonic, continuous, and strictly convex for every $i = 1, \ldots, I$. Then, every allocation in $C(\mathcal{E}_N)$ possesses the equal treatment property. As a result, core allocations of $\mathcal{E}_N$ can be represented as vectors in $\mathbb{R}_+^{IL}$, and $C(\mathcal{E}_N)$ is a subset of $\mathbb{R}_+^{IL}$, as are the Walrasian allocations.

Remark. Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy with continuous, strictly monotonic, and strictly convex preferences. Consider the following:

(i) The core of a replica economy diminishes as the number of replicas increases: $\forall N, C(\mathcal{E}_N) \supseteq C(\mathcal{E}_{N+1}) \supseteq C(\mathcal{E}_{N+2}) \supseteq \cdots$.

(ii) The equilibrium allocations of $\mathcal{E}_N$ can be represented as allocations in $\mathcal{E}$, i.e., the elements in $\mathcal{W}(\mathcal{E}_N)$ can be depicted in $\mathbb{R}_+^{IL}$.

(iii) $\mathcal{W}(\mathcal{E}_N) = \mathcal{W}(\mathcal{E})$ for every N.

(iv) An equilibrium allocation of $\mathcal{E}$ is part of the core of every replica economy $\mathcal{E}_N$: $\mathcal{W}(\mathcal{E}) \subseteq \bigcap_{N=1}^{\infty} C(\mathcal{E}_N)$.

Theorem 64. Debreu-Scarf Core Convergence Theorem. Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy where $\succeq_i$ is continuous, strictly monotonic, strictly convex, and $\omega_i > 0$ for all i. Also, assume $\overline{\omega} \geq 0$. If $x^* \in C(\mathcal{E}_N)$ for every $N \geq 1$, then $x^* \in \mathcal{W}(\mathcal{E})$. In other words,

$$\mathcal{W}(\mathcal{E}) = \bigcap_{N=1}^{\infty} C(\mathcal{E}_N).$$

Proof. See F. Echenique's lecture notes. The proof is not trivial. $\square$

The Debreu-Scarf Core Convergence Theorem provides significant insights into the behavior of competitive equilibria in large replicated economies. Here we outline the core ideas that underpin the theorem:

1. **Core of an Economy:** The core is defined as the set of allocations where no subgroup (coalition) of agents can rearrange their own resources among themselves to make everyone in the group better off, given the total resources available to them.

2. **Replication of the Economy:** The theorem considers an N-th replica of the original economy, creating multiple copies of each agent with identical preferences and endowments. As N increases, each agent's influence diminishes, approximating the conditions of perfect competition where agents are price-takers.

3. **Reduced Market Power:** In smaller economies, individuals or small groups can have significant market power. As the economy is replicated, this power is diluted because the relative influence of any single agent becomes negligible, pushing the economy toward a competitive equilibrium.

4. **Convergence to Walrasian Allocations:** In the limit, as the number of replicas becomes infinitely large, the core allocations converge to Walrasian allocations. This implies that competitive equilibrium, where market supplies meet demands at certain prices, is a likely outcome in very large economies.

5. **Economic Implications:** This convergence bridges cooperative game theory (the core) and non-cooperative market theory (competitive equilibria), demonstrating under what conditions these theoretical frameworks align and predict the same outcomes in large markets.

Conclusion: The Debreu-Scarf theorem illustrates why, in sufficiently large economies, competitive market theory is not just a theoretical ideal but an inevitable outcome of rational economic behavior.

8 Welfare and aggregation

We now shift our perspective from the market to that of a social planner and consider an exchange economy with production:

$$\mathcal{E} = \{\{\omega^i, \preceq_i\}_{i=1,\dots,I}, (Y^j)_{j=1,\dots,J}, \theta_{ij}\}.$$

Given the economy $\mathcal{E}$, we consider the set

$$X = \left\{(x^1, x^2, ..., x^I) \geq 0 : \sum_{i=1}^{I} x^i \in \overline{\omega} + Y\right\}.$$

The goal is to aggregate individual preferences to derive a "social preference".

Remark. Unfortunately, Arrow's Impossibility Theorem tells us that this aggregation cannot be conveniently done:

1. **Unrestricted domain:** considers the preferences of all individuals.
2. **Pareto efficiency:** if all individuals prefer allocation x over y, social preference must preserve this order.
3. **Non-dictatorial:** the aggregation should not always reflect the preference of any single individual.
4. **Independence of irrelevant alternatives:** the social preference between two options depends only on individual preferences regarding them.

Theorem 65. Arrow. There is no social aggregation rule that is unrestricted in domain, respects unanimity, is non-dictatorial, and maintains independence from irrelevant alternatives.

Due to Arrow's Theorem, some assumptions must be sacrificed in order to perform the aggregation. What is constructed is a **Social Welfare Function**. Specifically, we have the following definition.

Definition 66. Given an economy $\mathcal{E}$ and the corresponding set of feasible allocations X, we fix for each preference $\preceq_i$ a utility function u^i that represents it. A Social Welfare Function (SWF) is

$$\begin{aligned} W :\ & X \to \mathbb{R} \\ (x^1, ..., x^I) & \to W(u^1(x^1), ..., u^I(x^I)). \end{aligned}$$

Remark. In reality, a prior transformation is performed so that each point in the Edgeworth box corresponds to a point in $\mathbb{R}^I$ where each dimension corresponds to $u^i(x^i)$. Moreover, the set of Pareto optima corresponds to the frontier of the set formed in $\mathbb{R}^I$. Formally,

$$U = \{(u^1, ..., u^I) \in \mathbb{R}^I : \ \exists \ (x, y) \text{ feasible with } u^i \leq u^i(x^i), \ i = 1, ..., I\}.$$

Then,

$$W : U \to \mathbb{R}.$$

Definition 67. A SWF is increasing if $\forall \ u, u' \in U$, with $u \geq u'$, it holds that $W(u) \geq W(u')$ and if $u \gg u'$ then $W(u) > W(u')$. A SWF is strictly increasing if $u \geq u'$ and $u \neq u'$ imply that $W(u) > W(u')$.

1. Utilitarian: $W(u^1, ..., u^I) = \sum_{i=1}^I \beta_i u^i$. Indifferent to inequality.

2. Rawlsian: $W(u^1, ..., u^I) = \min_{i=1,...,I}\{\beta_i u^i\}$. Maximally averse to inequality.

3. CES (assuming $u^i \geq 0$)

$$W(u^1, ..., u^I) = \left(\sum_{i=1}^I (\beta_i u^i)^{1-\rho}\right)^{\frac{1}{1-\rho}}$$
$$W(u^1, ..., u^I) = \sum_{i=1}^I \beta_i \ln u^i, \ \rho = 1.$$

The parameter ρ measures indifference to inequality. $\rho = 0$ corresponds to the utilitarian and $\rho \to \infty$ to the Rawlsian.

Remark. The goal of the social planner is to solve

$$\max W(u^1, ..., u^I)$$
$$\text{s.a.}(u^1, ..., u^I) \in U.$$

On U, we can identify the Pareto Frontier

$$P = \{u \in U : \nexists u' \in U \text{ such that } \forall \ i : u'_i \geq u_i, \ \exists \ u'_i > u_i\}.$$

Proposition 68. An allocation (x, y) is Pareto optimal if and only if

$$(u^1(x^1), ..., u^I(x^I)) \in P.$$

Proposition 69. If the SWF W is increasing, the social planner's problem has a solution in P.

Next, we work with the utilitarian function.

Theorem 70. If U is convex

1. Given $\overline{u} \in P$, there exist $\beta_i \geq 0$, not all zero, such that the SWF $W(u^1, ..., u^I) = \sum_{i=1}^{I} \beta_i u^i$ reaches a maximum over U at $\overline{u}$.

2. Every utilitarian SWF reaches its maximum on U if it is convex.

The following are equivalent:

$$\begin{aligned} \max_{(u^1,...,u^I)} \; & \sum_{i=1}^{I} \beta_i u^i \\ \text{s.a. } & (u^1, ..., u^I) \in U \end{aligned}$$

and

$$\begin{aligned} \max_{(x^1,...,x^I)} \; & \sum_{i=1}^{I} \beta_i u^i(x^i) \\ \text{s.a. } & \sum_{i=1}^{I} x^i - \sum_{i=1}^{I} \omega^i = \sum_{j=1}^{J} y^j \\ & x^i \geq 0 \\ & y^j \in Y^j. \end{aligned}$$

If each Y^j is defined by $F^j(y^j) \leq 0$,

$$\begin{aligned} \max_{(x^1,...,x^I)} \; & \sum_{i=1}^{I} \beta_i u^i(x^i) \\ \text{s.a. } & \sum_{i=1}^{I} x^i - \sum_{i=1}^{I} \omega^i = \sum_{j=1}^{J} y^j \\ & x^i \geq 0 \\ & F^j(y^j) \leq 0. \end{aligned}$$

Assuming differentiability, concavity, and convexity, we conveniently obtain the optimality conditions

$$\frac{\partial_\ell u^i}{\partial_{\ell'} u^i} = \frac{\lambda_\ell}{\lambda_{\ell'}} = \frac{\partial_\ell F^j}{\partial_{\ell'} F^j}$$

for any combination of indices i, j, ℓ and ℓ'.

On the other hand, each consumer i solves

$$\max_{x^i \geq 0} u^i(x^i)$$
$$\text{s.a. } px^i \leq W^i$$

whose solution is characterized by

$$\frac{\partial_\ell u^i}{\partial_{\ell'} u^i} = \frac{p_\ell}{p_{\ell'}}.$$

Meanwhile, each firm j solves

$$\max py^j$$
$$\text{s.a. } F^j(y^j) \leq 0,$$

whose solution is characterized by

$$\frac{\partial_\ell F^j}{\partial_{\ell'} F^j} = \frac{p_\ell}{p_{\ell'}}.$$

Thus, the marginal rates are equal to the price ratios.

Remark. The coefficients β_i are the inverses of the multipliers γ^i such that

$$\partial_\ell u^i(x^i) = \gamma^i p_\ell.$$

8.1 Negishi's Method

1. Every Walrasian equilibrium is a Pareto optimum.

2. Every Pareto optimum can be solved as a problem of the type:

$$\begin{aligned} \max_{(x^1,\ldots,x^I)} \; & \sum_{i=1}^{I} \beta_i u^i(x^i) \\ \text{s.a. } & \sum_{i=1}^{I} x^i - \sum_{i=1}^{I} \omega^i = \sum_{j=1}^{J} y^j \\ & x^i \geq 0 \\ & F^j(y^j) \leq 0. \end{aligned}$$

3. Every Pareto optimum is a Walrasian equilibrium with transfers.

4. Every Walrasian equilibrium is a Walrasian equilibrium with zero transfers.

5. If the statements are valid, Negishi's method can be applied, which involves:

   a) Solving the problem for Pareto optima taking β_i as parameters.

   b) Calculating the prices according to β_i.

   c) Calculating transfers according to β_i.

   d) Finding the parameter set that makes the transfers zero.

   e) With the β_i found, determining the prices and allocations of equilibrium.

Let $(\succeq_i)_{i=1}^I$ be a collection of preferences where each $\succeq_i$ is represented by a utility function $u_i : \mathbb{R}_+^L \to \mathbb{R}$. Consider an economy identified by a vector of endowments $\omega \in \mathbb{R}_+^{IL}$, structured such that for each $\omega \in \mathbb{R}_+^{IL}$, $(\succeq_i, \omega_i)_{i=1}^I$. An economy has a fixed structure of endowments if there exists $\alpha = (\alpha_i)_{i=1}^I \in \mathbb{R}_+^I$ with $\sum_{i=1}^I \alpha_i = 1$ and $\omega_i = \alpha_i \overline{\omega}$.

Theorem 71. Eisenberg's Theorem. Assume that each $\succeq_i$ is represented by a continuous and homogenous degree one utility function u_i. Then the aggregate demand of the economy is generated by a representative consumer, whose utility function $U : \mathbb{R}_+^L \to \mathbb{R}$ is given by:

$$U(x) = \max_{(x_1,\ldots,x_I)\in\mathbb{R}_+^{IL}} \left(\sum_{i=1}^{I} (u_i(x_i))^{\alpha_i} \right) \quad \text{s.t.} \quad x = \sum_{i=1}^{I} x_i$$

9 Conclusion

Throughout these lecture notes, we have extensively explored the foundational and complex aspects of general equilibrium theory, from basic 2×2 economy models to more sophisticated scenarios involving production and pure exchanges. These discussions have been vital in understanding resource allocation and efficiency within various economic frameworks.

While studying models with infinitely many goods might not be at the cutting edge of new economic research, they offer crucial insights and are vital for deepening our understanding of theoretical economics. These models require robust mathematical tools, specifically from topology and functional analysis, to tackle their complexities effectively. For those interested in delving further into this area, the work by Araujo and Klinger Monteiro, provides foundational insights (Aloisio Araujo & Paulo Klinger Monteiro 1992).