Dynamic Programming

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1 Introduction

The following model introduces the main theoretical results of the matter from an economic point of view. We will consider identical and infinitely lived household, $t \in \mathbb{Z}_+$, a consumption good y_t which is produced using capital stock and labor

$$y_t = F(k_k, \ell_t),$$

and we denote consumption by c_t and investment by i_t [Lucas et al. (1989)]. Since there are no exports, imports, government expenditure, or other kind of economic contribution source,

$$c_t + i_t \leq y_t.$$

This consumption-saving decision is the only allocation decision the economy must make.

Now, capital stock is assumed to depreciate at a constant rate $0 < \delta < 1$. Thus, since investment is used to acquire more capital,

$$k_{t+1} = (1-\delta)k_t + i_t.$$

We will consider that labour is supplied exogenously. Finally, since current consumption is preferred over future consumption, the optimization problem which is considered is

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t U(c_t) \\
\text{s.t. :}\, &k_{t+1} = (1-\delta)k_t + i_t \\
&c_t + i_t \leq y_t \\
&y_t = F(k_t, \ell_t) \\
&k(0) = k_0.
\end{aligned}$$

Here the discount term is $\beta \in (0,1)$.

1.1 Preliminaries

Definition 1. The production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is class C^1, strictly increasing, homogeneous of degree-one and strictly concave

$$
\begin{aligned}
F(0,\ell) &= 0\\
\frac{\partial F(k,\ell)}{\partial k} &> 0\\
\frac{\partial F(k,\ell)}{\partial \ell} &> 0\\
\lim_{k\to 0}\frac{\partial F(k,1)}{\partial k} &= +\infty\\
\lim_{k\to \infty}\frac{\partial F(k,1)}{\partial k} &= 0.
\end{aligned}
$$

There properties are studied with much more detail in Barro and i Martin (2003).

Remark. Since $i_t = k_{t+1} - (1-\delta)k_t$,

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, \ell_t).$$

Definition 2. The objective function of the multi-stage optimization problem $\mathcal{P}_D$ is

$$u(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t U(c_t).$$

This definition is explained in [Mas-Colell et al. (1995)], [Varian (1992)] or [Gravelle and Rees (2004)]. Here $U : \mathbb{R}_+ \to \mathbb{R}$ is bounded, continuously differentiable, strictly increasing and strictly concave. Furthermore,

$$\lim_{c\to 0} U'(c) = \infty.$$

Remark. The objective is to choose the sequence $\{c_t, k_{t+1}, \ell_t\}_{t=0}^{\infty}$.

Remark. Considering the equality

$$c_t + i_t = F(k_t, \ell_t) \triangleq F(k_t, 1)$$

and the dynamic for k_t, we have, defining $f(k_t) = F(k_t, 1) + (1-\delta)k_t$, that

$$c_t = f(k_t) - k_{t+1}.$$

Thus, the $\mathcal{P}_D$ is rewritten as follows

$$
\begin{aligned}
\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[f(k_t) - k_{t+1}]\\
\text{s.t.:} 0 \leq k_{t+1} \leq f(k_t)\\
k_0 < 0,\ \text{given.}
\end{aligned}
$$

Remark. Although ultimately we are interested in the case where the planning horizon is infinite, it is possible to have problems of the kind

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t U[f(k_t) - k_{t+1}]$$
$$\text{s.t.:} 0 \leq k_{t+1} \leq f(k_t)$$
$$k_0 < 0, \text{ given.}$$

In such a case, it would only be a standard concave programming problem where $\{k_t\}_{t=0}^{T} \subset \mathbb{R}^{T+1}$, is a convex, closed and bounded set. Here the solution is completely characterized by Kuhn-Tucker conditions Acemoglu (2009)

$$\beta f'(k_t) U'(f(k_t) - k_{t+1}) = U'(f(k_{t-1}) - k_t), \ t = 1, ..., T$$
$$k_{T+1} = 0.$$

Indeed,

$$\frac{d}{dk_t}\left(\sum_{t=0}^{T} \beta^t U[f(k_t) - k_{t+1}]\right) = 0$$

implies

$$\beta^t f'(k_t) U'[f(k_t) - k_{t+1}] + \beta^{t-1}(-1) U'[f(k_{t-1}) - k_t] = 0.$$

This is,

$$\beta f'(k_t) U'[f(k_t) - k_{t+1}] = U'[f(k_{t-1}) - k_t]. \tag{1}$$

Since $0 \leq k_{t+1} \leq f(k_t)$, $U'(0) = \infty$,

$$k_{T+1} = 0.$$

Example 3. Let $f(k) = k^\alpha$, $0 < \alpha < 1$ and $U(c) = \ln(c)$. From (1), we obtain

$$\beta \alpha k_t^{\alpha-1}\left(\frac{1}{k_t^\alpha - k_{t+1}}\right) = \left(\frac{1}{k_{t-1}^\alpha - k_t}\right). \tag{2}$$

Let $z_t = k_t / k_{t-1}^\alpha$. Thus, (2) becomes

$$\beta \alpha k_t^{\alpha-1}\left(\frac{1}{k_t^\alpha(1 - z_{t+1})}\right) = \left(\frac{1}{k_{t-1}^\alpha(1 - z_t)}\right). \tag{3}$$

Simplifying (3),

$$\beta \alpha k_t^{-1}(1 - z_t) = \frac{1 - z_{t+1}}{k_{t-1}^\alpha}. \tag{4}$$

$$z_{t+1} = 1 - \frac{\beta \alpha (1 - z_t)}{z_t}. \tag{5}$$

Finally, by induction,

$$z_t = \alpha\beta\left(\frac{1 - (\alpha\beta)^{T-t+1}}{1 - (\alpha\beta)^{T-t+2}}\right)$$
$$k_{t+1} = \alpha\beta\left(\frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}\right) k_t^\alpha, \ t = 0, 1, ..., T.$$

Remark. What happens if we take $T \to \infty$? $k_{t+1} = \alpha\beta k_t^\alpha$. Is this the solution? Actually, it is not as easy as take $T \to \infty$. We will now focus in the infinite horizon case. For this, we will start afresh.

1.2 General case

The problem that faces the planner in period $t = 0$ is that of choosing today's consumption (c_0), and tomorrow's beginning-of-period capital k_1, nothing else. If the preferences of the planer, over c_0 and k_1, are known, we could simply maximize over (c_0, k_1). Assume that $\mathcal{P}_D$ is already solved, with $k_{t+1} = g(k_t)$. Then, we could define $v : \mathbb{R}_+ \to \mathbb{R}$ by taking $v(k_0)$ to be the value of the maximized objective function in $\mathcal{P}_D$. This function is known as *value function.* With v so defined, $v(k_1)$ would give the value of the utility from period $t = 1$ on that could be obtained with $k(t_1) = k_1$ and $\beta v(k_1)$ would be the value of this utility discounted bak to period $t = 0$. Then, the planner's problem $\tilde{P}_D$ would be

$$\begin{aligned} &\max_{c_0, k_1} U(c_0) + \beta v(k_1) \\ &\text{ s.t. } c_0 + k_1 \le f(k_0) \\ &\qquad c_0, k_1 \ge 0,\ \ k_0 > 0,\ \text{given.} \end{aligned}$$

If the function v were known, we could use to define a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ as follows, for each $k_0 \ge 0$, let $k_1 = g(k_0)$ and $c_0 = f(k_0) - g(k_0)$ be the functions that attain the maximum in $\tilde{P}_D$.

Remark. If $v(k_0)$ solves $\mathcal{P}_D$,

$$v(k_0) = \max_{0 \le k_1 \le f(k_0)} \{U[f(k_0) - k_1] + \beta v(k_1)\}.$$

Notice that when the problem is looked at in this recursive way, we can simply rewrite

$$v(k) = \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta v(y)\}. \tag{6}$$

Moreover, (6) is a functional equation and the problem which is faced is called a *dynamic programming* problem. If we knew that v was differentiable and that the maximizing value of y was interior, then by F.O.C,

$$\begin{aligned} U'[f(k) - g(k)] &= \beta v'[g(k)] \\ v'(k) &= f'(k)U'[f(k) - g(k)]. \end{aligned}$$

The first equation equates the marginal utility of consuming current output to the marginal utility of allocating it to capital and enjoying augmented consumption next period. The second equation states that the marginal value of current capital and the marginal utility of using capital stock in current production and allocating its return to current consumption.

2 Existence and uniqueness

Recall that $\mathcal{P}_D$ leads to (6). Our purpose now is to prove the existence and uniqueness of a function v satisfying (6), and to derive its properties. For this, we will define the following sequence

$$v_{n+1}(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v_n(y)\}, \ n = 0, 1, 2, ... \tag{7}$$

How to prove that there exists v such that $v_n \to v$? The idea is to use a fixed point argument [Ok (2007)]. Notice that the sequence $\{v_n\}_{n \in \mathbb{N}}$ is increasing [Lucas et al. (1989)].

2.1 Some preliminaries from Real Analysis

Definition 4. A real vector space V is a set of vectors together with two operations, addition and scalar multiplications. For any two vectors $x, y \in V$, $x + y \in V$, and for every $x \in V$ and $\alpha \in \mathbb{R}$, $\alpha x \in V$. These operations must obey some algebraic rules that we state next right below. Given $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. $\alpha(x + y) = \alpha x + \alpha y$
4. $(\alpha + \beta)x = \alpha x + \beta y$
5. $(\alpha\beta)x = \alpha(\beta x)$
6. $\exists\ \theta \in V$ (zero vector) such that $x + \theta = x$
7. $0x = \theta$
8. $1x = x$.

Example 5. The sets $\mathbb{R}^L, \mathbb{C}^L$, $X = \{x \in \mathbb{R}^2 : \ x = az, \ a \in \mathbb{R}\}$, $z \in \mathbb{R}^2$, $C^0([0, 1], \mathbb{R})$ are real vector spaces, while $\mathbb{Z}$ and $\mathbb{S}^1$ are not.

Definition 6. A metric space is a set X with a metric (distance function), $d : X \times X \to \mathbb{R}$ such that, for any $x, y, z \in X$

1. $d(x, y) \geq 0$, with equality if and only if $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$.

We usually denote (X, d).

Example 7. The sets

- $(\mathbb{Z}, |x-y|)$
- $C^0([a,b], \uparrow, \max_{a \leq t \leq b} |x(t)-y(t)|)$
- $C^0\left([a,b], \uparrow, \int_a^b |x(t)-y(t)|dt\right)$

are metric spaces.

Definition 8. A real vector normed space is a vector space X together with a norm $||\cdot|| : X \to \mathbb{R}$ which satisfies the following properties. Given $x, y \in S$ y $\alpha \in \mathbb{R}$,

1. $||x|| \geq 0$ with equality if and only if $x = \theta$.
2. $||\alpha x|| = |\alpha| \cdot ||x||$
3. $||x+y|| \leq ||x|| + ||y||$ (triangle inequality).

Example 9. The following are norms

1. $||x||_2 = \sqrt{\sum_{i=1}^L x_i^2}$, $X = \mathbb{R}^L$
2. $||x||_{\max} = \max_{1 \leq i \leq L}\{|x_i|\}$, $X = \mathbb{R}^L$
3. $||x||_1 = \sum_{i=1}^L |x_i|$, $X = \mathbb{R}^L$
4. $||f||_1 = \int_a^b |f(t)|dt$, $X = C^0([a,b], \mathbb{R})$
5. $||f||_\infty = \sup_{a \leq t \leq b} |f(t)|$, $X = C^0([a,b], \mathbb{R})$.

Definition 10. A sequence $\{x\}_{n \geq 0}$ in (X, d) converges to $x \in X$ if for each $\epsilon > 0$, there exists N_ϵ such that

$$d(x_n, x) < \epsilon, \ n > N_\epsilon.$$

Remark. *If we want to verify the convergence it is needed to have a candidate for the limit point x (which is, by the way, in $\mathbb{R}^L$, unique[1]). When a candidate is not immediately available, the following result is often useful.*

Definition 11. A sequence $(\{x_n\}_{n=0}^\infty) \subset X$ is a Cauchy sequence if for every $\epsilon > 0$, there exists N_ϵ such that

$$d(x_n, x_m) < \epsilon, \ n, m \geq N_\epsilon.$$

Proposition 1. *A Cauchy sequence is convergent and a convergent sequence is a Cauchy sequence.*

Definition 12. A metric space (X, d) is complete if every Cauchy sequence in X converges to an element in X.

[1] In every Hausdorff space.

Example 13. $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Definition 14. A Banach space is a complete normed space.

Lemma 15. *Let $X \subset \mathbb{R}^L$, and let $C(X)$ be the set of bounded continuous functions $f : X \to \mathbb{R}$ with sup norm $||f||_\infty = \sup_{x\in X} |f(x)|$. Then, $C(X)$ is a normed vector space.*

Proof. The fact that is a normed vector space is by definition. To see that $||\cdot||_\infty$ defined a norm, first, notice that

$$0 \leq \sup_{x\in X} |f(x)| = 0 \Rightarrow f(x) = 0,\ \forall\ x \in X.$$

By continuity, $f = \theta$. Then, trivially $||f||_\infty \geq 0$. Then, given $\alpha \in \mathbb{R}$

$$||\alpha f||_\infty = \sup_{x\in X} |\alpha f(x)| = \sup_{x\in X} |\alpha| \cdot |f(x)| = \alpha \sup_{x\in X} |f(x)| = |\alpha| \cdot ||f||_\infty.$$

Finally, the triangle inequality follows from the usual triangle inequality for $|\cdot|$,

$$|f + g| \leq |f| + |g|.$$

Since this is for every $x \in X$, $|f| + |g|$ is upper bound for $|f + g|$. Thus, by the definition of the supremum,

$$||f + g||_\infty = \sup_{x\in X} |f + g| \leq |f| + |g| \leq \sup_{x\in X} |f| + \sup_{x\in X} |g| = ||f||_\infty + ||g||_\infty.$$

In all moment it is used that $||\varphi||_\infty < \infty$. $\square$

Theorem 16. *The set $C(X)$ is a complete normed vector space.*

The proof is given in Lucas et al. (1989). Now, we are able to address the main result which is needed to study our problem (7), the Contraction Mapping Theorem.

Definition 17. Let (X, d) be a metric space and $T : X \to X$ be a function mapping X to itself. Then, T is a contraction mapping with modulo $\theta \in (0, 1)$ if $d(Tx, Ty) \leq \theta d(x, y)$, $\forall\ x, y \in X$.

Definition 18. A fixed point $x \in X$ is such that $Tx = x$.

Theorem 19. ***(Contraction Mapping Theorem.)*** *If (X, d) is a complete metric space and $T : X \to X$ a contraction mapping with modulus θ, then*

1. *T has exactly one fixed point $x^* \in X$.*
2. *For any $x_0 \in X$, $d(T^n x_0, x^*) \leq \theta^n d(x_0, x^*)$.*

Proof. Take $x_0 \in X$ and define $\{x_n\}_{n=0}^\infty$ recursively by $x_{n+1} = Tx_n$ so that

$x_n = T^n x_0$. Then, by the contraction property,

$$\begin{aligned} d(x_2, x_1) &= d(Tx_1, Tx_0) \leq \theta d(x_0, x_1) \\ d(x_3, x_2) &= d(Tx_2, Tx_1) \leq \theta d(x_2, x_1) \leq \theta^2 d(x_0, x_1) \\ &\vdots \\ d(x_{n+1}, x_n) &\leq \theta^n d(x_1, x_0). \end{aligned}$$

For any $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + ... + d(x_{n+1}, x_n) \\ &\leq \theta^m d(x_1, x_0) + ... + \theta^n d(x_1, x_0) \\ &= \left(\sum_{k=n}^{m} \theta^k\right) d(x_1, x_0) \\ &\leq \frac{\theta^n}{1-\theta} d(x_1, x_0). \end{aligned}$$

Since X is complete, it follows that $x_n \to x^*$ for some $x^* \in X$. Furthermore,

$$d(x^*, Tx^*) \leq d(Tx^*, T^n x_0) + d(T^n x_0, x^*) \leq \theta d(x^*, T^{n-1}x_0) + d(T^n x_0, x^*) \to 0.$$

Finally, to show the uniqueness, assume there exists $y \in X$, $y \neq x^*$ s.t. $Ty = y$. Then,

$$0 < a = d(x^*, y) = d(Tx^*, Ty) \leq \theta d(x^*, y) = \theta a \Rightarrow\Leftarrow.$$

□

Theorem 20. ***(Blackwell).*** *Let $X \subset \mathbb{R}^L$ and $B(X)$ be the set of all bounded functions from X to $\mathbb{R}$, with respect to $||\cdot||_\infty$. Then, let $T : B(X) \to B(X)$ be an operator satisfying*

1. *Monotonicity: $f, g \in B(X)$, $f(x) \leq g(x) \Rightarrow T(f(x)) \leq T(g(x))$.*
2. *There exists $\beta \in (0, 1)$ such that*

$$T(f + a)(x) \leq Tf(x) + \beta a, \ (f + a)(x) = f(x) + a$$

Then T is a contraction.

Proof. If $f(x) \leq g(x)$ and

$$T(f)(x) \leq T(g)(x),$$

then, since $f \leq g + ||f - g||$

$$Tf \leq T(g + ||f - g||) \leq T(g) + \beta||f - g||.$$

Thus,

$$||Tf - Tg|| \leq \beta||f - g||.$$

□

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