Exercise 6.3.8. Consider the $\mathcal{P}_c$ problem

$$\begin{aligned} J^* &= \max_{u(t)\in\Omega} \int_{t_0}^{t_1} F(x,u,t)\,dt \\ \text{s.t.}: \qquad x' &= f(x,u,t), \\ x(t_0) &= x_0, \\ x(t_1) &= x_1 \ . \end{aligned}$$

where q is the extraction rate of a resource of stock x, and $U(q)$ is the gross consumer benefit

$$U(q) = \int_0^q D(s)\,ds,$$

with $p = D(s)$ the inverse demand function. Prove that[22]

$$\delta = \frac{1}{p(t)}\frac{dp}{dt}.$$

<u>Hint:</u> The Hamiltonian does not depend on x, so λ is constant. Combine this with the maximum condition and $U'(q) = D(q) = p$ to show that $e^{-\delta t}p(t)$ is constant, then differentiate.

6.4 Infinite horizon

The preceding sections offer a self-contained introduction to optimal control problems in continuous time over a finite horizon. Several

[22] This condition, known as Hotelling's rule, states that the price of the resource must rise at the discount rate δ so that the present value of the resource remains constant over time. If a firm holds a valuable resource, selling some today yields cash now; selling later requires the future price to compensate for the delay. Hotelling's rule guarantees that, in present-value terms, selling now or later yields the same return.

economic problems are naturally posed in such a framework: the duration of a harvest season, the operational lifespan of a machine, the lifetime of an individual. Most modern growth models, however, require an infinite-horizon formulation. This is justified on two grounds. First, the capital-accumulation process has no natural terminal date. Second, although individuals die, society persists through the succession of its members.

This section studies the limiting case "$t_1 = \infty$", where all variables are scalar. The goal is a self-contained and usable theory: to formulate the optimal control problem, to identify the new phenomena that an unbounded horizon introduces, and to assemble a workable set of necessary and sufficient conditions for solving the economic models of the chapter, among them the Ramsey–Cass–Koopmans model, the AK and Lucas models, and Tobin's Q. We proceed in the order in which the questions naturally arise. After formulating the problem, we examine the difficulties peculiar to the infinite horizon and the sense in which an optimum exists; we then state the necessary conditions (the Maximum Principle) and supply the boundary condition at infinity (the transversality condition). Specializing to the discounted case through the current-value Hamiltonian casts these conditions in the form used throughout the applications, and a final set of sufficient conditions certifies that a candidate path is genuinely optimal.

To keep the exposition focused on intuition and on the tools strictly needed to address the economic applications of the chapter, the more technical results, together with all proofs, are deferred to Appendix 6.A.3. It further contains an application to contract theory, the principal-agent screening problem (Appendix 6.A.1), whose formulation requires a substantial modeling apparatus and is therefore presented separately rather than as an in-text example.

FORMULATION OF THE PROBLEM

In this section we study the infinite-horizon problem $\mathcal{P}_c^\infty$,

$$\begin{aligned} \max_u \qquad & J = \int_{t_0}^{\infty} F(x(t), u(t), t)\, dt \\ \text{s.t.:} \qquad & x'(t) = f(x(t), u(t), t), \quad t \ge t_0, \\ & x(t_0) = x_0, \\ & u(t) \in \Omega, \quad t \ge t_0, \\ & \lim_{t\to\infty} \beta(t)\, x(t) \ge x_1, \end{aligned} \tag{6.4.1}$$

where $x_1 \in \mathbb{R}$, the control region $\Omega \subseteq \mathbb{R}$ is a non-empty closed convex set,[23] and the weight $\beta : [t_0, \infty[\to \mathbb{R}$ is continuous with $\lim_{t\to\infty} \beta(t) = \gamma$ for some $\gamma \in [0, \infty[$. The standing assumptions are:

(A1) $F, f : \mathbb{R} \times \Omega \times [t_0, \infty[\to \mathbb{R}$ are of class C^1.

(A2) For every piecewise-continuous control $u : [t_0, \infty[\to \Omega$, the initial value problem $x'(t) = f(x(t), u(t), t)$, $x(t_0) = x_0$, has a unique solution on $[t_0, \infty[$, denoted x_u.

(A3) For every piecewise-continuous control $u : [t_0, \infty[\to \Omega$, the integral J in (6.4.1), evaluated along (x_u, u), is well defined; in particular it is finite along any optimal pair.

[23] As in the finite-horizon problems of the previous sections, the control region Ω is often kept fixed. The theory extends to time-varying regions $\Omega(t)$ and state-dependent regions $\Omega(x(t))$, see Hartl et al. (1995).

By (A2), every piecewise-continuous control with values in Ω generates a unique state trajectory x_u, which lets us collect the admissible controls in the class

$$\mathcal{U} = \left\{ u : [t_0, \infty[\to \Omega \text{ piecewise continuous } : \lim_{t\to\infty} \beta(t)\, x_u(t) \geq x_1 \right\}.$$

A pair (x_u, u) with $u \in \mathcal{U}$ is called an admissible pair, and the problem maximizes J over $u \in \mathcal{U}$. The control region constrains u pointwise, through $u(t) \in \Omega$, whereas the terminal condition constrains the entire trajectory; the class $\mathcal{U}$ encodes both requirements.

The weight β makes the terminal condition flexible enough to cover, in a single formulation, the cases arising in the applications. Taking $\beta = 0$ identically and $x_1 = 0$ leaves the terminal state free; taking $\beta = 1$ identically imposes an asymptotic floor on the state, such as $\lim\limits_{t\to\infty} k(t) \geq 0$ for a capital stock; and taking $\beta(t) = e^{-\int_{t_0}^{t} r(s)\, ds}$ with $x_1 = 0$ yields

$$\lim_{t\to\infty} e^{-\int_{t_0}^{t} r(s)\, ds}\, a(t) \;\geq\; 0,$$

the no-Ponzi-game condition[24] on the asset holdings $a(t)$ of a household that borrows at the interest rate $r(t)$. Writing the household's debt as $-a(t)$, the factor $e^{-\int_{t_0}^{t} r(s)\, ds}$ discounts it at exactly the rate at which interest accrues, so the condition forces the discounted debt to vanish asymptotically: debt may not grow

[24] Charles Ponzi (1882–1949) ran a 1920 fraud that paid earlier investors with money from later ones, the prototype of the pyramid scheme, solvent only while it keeps expanding. The condition forbids the analogous strategy of servicing debt by issuing ever more debt.

as fast as the interest it generates. This is precisely the prohibition on paying interest by borrowing ever more. When r is bounded away from zero, $\beta(t) \to 0$, so $\gamma = 0$.

Remark 6.4.1. (Difficulties of the infinite horizon). Passing from a bounded horizon $[t_0, t_1]$ to $t_1 = \infty$ raises five issues absent in the finite-horizon theory.

1. Convergence of the objective: along any admissible pair the integrand $t \mapsto F(x(t), u(t), t)$ is continuous, or piecewise continuous with one-sided limits at the finitely many discontinuities of the control, hence integrable on every compact interval, so the finite-horizon integral $\int_{t_0}^{T} F(x(t), u(t), t)\, dt$ is finite. The new difficulty is the convergence of the improper integral
$$J = \lim_{T\to\infty} \int_{t_0}^{T} F(x(t), u(t), t)\, dt,$$
which need not exist and must be checked case by case (a sufficient condition is given in Remark 6.4.2).

2. Existence of an optimum: the admissible class $\mathcal{U}$ is infinite-dimensional, so the supremum of the objective over $\mathcal{U}$ need not be attained, and need not be finite. We do not address the existence of a maximizer, which relies on tools not self-contained in this book. Rigorously one writes sup throughout the chapter, but since an optimum exists in every application we write max, as in (6.4.1).

3. Terminal condition: along a general admissible trajectory $\beta(t)\, x(t)$ may fail to converge, so the rigorous terminal

condition uses the limit inferior, $\liminf_{t\to\infty} \beta(t)\, x(t) \geq x_1$, requiring the trajectory to remain eventually above every level below x_1. When the limit exists, as in the applications, this coincides with the lim used in (6.4.1).

4. Boundary condition at infinity: the terminal-state (transversality) conditions of Section 6.3 do not extend to the infinite horizon by simply taking limits;[25] supplying the correct condition is the object of the transversality theory developed below.

5. Continuity of the optimal control: the necessary conditions are stated for piecewise-continuous controls, but the applications differentiate the first-order condition in time, which presupposes a control without jumps. Under strict concavity and compactness hypotheses the optimal control is in fact continuous; this is taken up after the sufficient conditions.

Appendix 6.A.3 develops some of these points; we now turn to the necessary and transversality conditions for the $\mathcal{P}_c^\infty$ problem.

Remark 6.4.2. A frequent specification is $F(x,u,t) = e^{-\rho t}\, G(x,u,t)$ with $\rho > 0$. If there exists $M > 0$ such that $|G(x,u,t)| \leq M$ for every admissible pair (x,u) and every $t \geq t_0$, then

$$\left| \int_{t_0}^{\infty} F(x,u,t)\, dt \right| \;\leq\; M \int_{t_0}^{\infty} e^{-\rho t}\, dt \;=\; \frac{M\, e^{-\rho t_0}}{\rho} \;<\; \infty,$$

so J is well defined. The bound on G rather than on F is what is needed: from $|F(x,u,t)| \leq M$ alone one obtains only $\int_{t_0}^{T} |F(x,u,t)|\, dt \leq M\,(T - t_0)$, whose limit as $T \to \infty$ is infinite.

[25] For instance, $\lambda(t_1) = 0$ does not, in general, imply $\lim_{t\to\infty} \lambda(t) = 0$.

NECESSARY CONDITIONS: THE MAXIMUM PRINCIPLE FOR THE $\mathcal{P}_c^\infty$ PROBLEM

The first task in solving any model is to produce candidate optimal paths. The infinite-horizon Maximum Principle does exactly this, extending the necessary conditions of the finite-horizon theory to the $\mathcal{P}_c^\infty$ problem.

Theorem 6.4.1. (Infinite-horizon Maximum Principle). Under (A1)–(A3), let (x^*, u^*) solve the $\mathcal{P}_c^\infty$ problem with u^* piecewise continuous. Then there exists an absolutely continuous costate $\lambda^* : [t_0, \infty[\to \mathbb{R}$ such that, for almost every $t \geq t_0$, the pointwise maximum condition

$$\max_{v\in\Omega} H\left(x^*(t), v, \lambda^*(t), t\right) \; = \; H\left(x^*(t), u^*(t), \lambda^*(t), t\right) \qquad (6.4.2)$$

holds, and at every point of continuity of u^* the canonical equations

$$(\lambda^*)'(t) \; = \; -\frac{\partial H}{\partial x}\left(x^*(t), u^*(t), \lambda^*(t), t\right), \qquad (6.4.3)$$

$$(x^*)'(t) \; = \; \frac{\partial H}{\partial \lambda}\left(x^*(t), u^*(t), \lambda^*(t), t\right) \qquad (6.4.4)$$

are satisfied, together with $x^*(t_0) = x_0$ and $\lim_{t\to\infty} \beta(t)\, x^*(t) \geq x_1$. If, in addition, $u^*(t)$ is an interior point of Ω for almost every t, then the pointwise maximum condition implies the interior first-order condition

$$\frac{\partial H}{\partial u}\left(x^*(t), u^*(t), \lambda^*(t), t\right) \; = \; 0. \qquad (6.4.5)$$

Two comments help read the statement. First, the maximum condition (6.4.2), the canonical equations (6.4.3)–(6.4.4), and the interior first-order condition (6.4.5) are formally identical to their finite-horizon counterparts (Theorem 6.1.1). This is no accident: optimality forces the Hamiltonian to be maximized in u at each instant, a pointwise property insensitive to the length of the horizon, so the conditions carry over to $\mathcal{P}_c^\infty$ unchanged. In practice, the first-order condition (6.4.5) determines the optimal control as a function of the state and the costate, $u^*(t) = u(x^*(t), \lambda^*(t), t)$; substituting this expression into the canonical equations (6.4.3)–(6.4.4) eliminates the control and leaves two differential equations in the two unknowns x^* and λ^*. This planar system, is the object actually studied in the applications, through the phase-diagram methods of the dynamical-systems chapters.

Second, exactly one ingredient of the finite-horizon theory does not survive the passage to $t_1 = \infty$: the boundary condition on the costate. Over a finite horizon the transversality condition fixes $\lambda^*(t_1)$, for instance $\lambda^*(t_1) = 0$ when the terminal state is free; over an infinite horizon there is no terminal date, and the naive replacement $\lim\limits_{t\to\infty} \lambda^*(t) = 0$ may fail (Example 6.4.2). The canonical system then has a whole family of solutions, and a condition at infinity is needed to single out the optimum. Theorem 6.4.1 is silent on this point, imposing no restriction on the costate at infinity; supplying the correct condition is the object of the transversality theory developed next.

THE TRANSVERSALITY CONDITION FOR THE $\mathcal{P}_c^\infty$ PROBLEM

The Maximum Principle determines the optimal control through the canonical equations, but it says nothing about the costate as $t \to \infty$, and this is a real gap. After the optimal control is substituted in, the canonical equations form a planar system, whose general solution depends on two free constants. The initial condition $x^*(t_0) = x_0$ fixes one of them; a second condition is needed to select the optimal trajectory from the remaining one-parameter family. Over a finite horizon that second condition is the transversality condition at the terminal time t_1; over an infinite horizon there is no terminal time, so we must supply a replacement, a boundary condition at infinity.

To state that condition we first introduce the value function, which records the best payoff the problem can deliver starting from an arbitrary state and time. For $(\xi, t) \in \mathbb{R} \times [t_0, \infty[$, define

$$J(\xi, t) \;=\; \max_{u \in \mathcal{U}_{t,\xi}} \int_t^{\infty} F(x(s), u(s), s)\, ds, \tag{6.4.6}$$

where $\mathcal{U}_{t,\xi}$ is the admissible class $\mathcal{U}$ with the horizon $[t_0, \infty[$ replaced by $[t, \infty[$ and the initial condition $x(t_0) = x_0$ replaced by $x(t) = \xi$ (the maximum is read as a supremum when no maximizer exists). In words, $J(\xi, t)$ is the highest value attainable if the system starts in state ξ at time t and is controlled optimally from then on. In particular, $J(x_0, t_0)$ is the optimal value of $\mathcal{P}_c^{\infty}$. The value function has two properties that drive the analysis: the Principle of Optimality, which says that an optimal plan remains optimal when restarted from any state it reaches, and the Hamilton–Jacobi–Bellman equation. Both are established in Appendix 6.A.3; here we need only the boundary condition they yield.

Theorem 6.4.2. (Transversality condition: undiscounted case). Consider problem $\mathcal{P}_c^{\infty}$ under (A1)–(A3), and let u^* be a piecewise-continuous interior optimal control. Assume there exists $T \geq t_0$ such that the value function J in (6.4.6) is of class C^1 in a neighborhood of the set $\{(x^*(t), t) : t \geq T\}$ and satisfies

$$\lim_{t\to\infty} \frac{\partial J}{\partial t}(x^*(t), t) \;=\; 0. \tag{6.4.7}$$

Then (x^*, u^*, λ^*) satisfies the conditions of Theorem 6.4.1 and, in addition,

$$\lim_{t\to\infty} H\left(x^*(t), u^*(t), \lambda^*(t), t\right) \;=\; 0. \tag{6.4.8}$$

Two points deserve attention. The hypotheses require that, for t large enough (beyond some $T \geq t_0$), the value function be continuously differentiable (C^1) along the optimal path and that its partial derivative with respect to time vanish at infinity; in economic terms, the date itself eventually ceases to affect the optimal continuation value. The conclusion is the transversality condition, whose form is the essential point: it restricts the Hamiltonian, not the costate. The two restrictions one might expect instead, $\lim\limits_{t\to\infty} \lambda^*(t) = 0$ and $\lim\limits_{t\to\infty} \lambda^*(t)\, x^*(t) = 0$, can both fail at a genuine optimum, as Halkin's counterexample (Example 6.4.2) shows. What always holds is $\lim\limits_{t\to\infty} H = 0$: along the optimal path the Hamiltonian is the total value generated per unit time, the current payoff plus the shadow value of the change in the state, and at infinity this flow must vanish. The proof rests on the Principle of Optimality, the Hamilton–Jacobi–Bellman equation, and the identification of the costate with the shadow price of the state, $\lambda^*(t) = \partial J/\partial\xi\,(x^*(t), t)$; it is given in Appendix 6.A.3 (Proof of Theorem 6.4.2).

The following example, adapted from Acemoglu (2009), illustrates Theorem 6.4.2 in a familiar growth setting: the costate does not vanish at infinity, but the Hamiltonian does.

Example

Example 6.4.1. Consider a simple optimal growth model, used here to illustrate the undiscounted transversality condition. A planner chooses the consumption path $c(t) > 0$ of a representative household so as to maximize its cumulative logarithmic utility. Output is produced from the capital stock $k(t)$ by the Cobb–Douglas technology k^α, with $\alpha \in]0, 1[$, so capital has decreasing returns; capital depreciates at the rate $\delta > 0$ and accumulates through the part of output that is not consumed, which gives the law of motion $k' = k^\alpha - \delta k - c$. The economy starts from $k(0) = 1$, and the capital stock is required to remain asymptotically non-negative, $\lim_{t\to\infty} k(t) \geq 0$. Unusually, the objective carries no discount factor; convergence of the integral is secured instead by measuring instantaneous utility relative to its steady-state value $\log c^*$, so that the integrand vanishes at the steady state. The problem is

$$
\begin{aligned}
\max_{c(t)\in\mathbb{R}_{++}} \qquad & J = \int_0^\infty (\log c(t) - \log c^*)\, dt \\
\text{s. t. :} \qquad & k'(t) = k(t)^\alpha - \delta\, k(t) - c(t), \\
& k(0) = 1, \\
& \lim_{t\to\infty} k(t) \geq 0,
\end{aligned}
$$

where the steady-state pair (k^*, c^*) is given by[26]

$$k^* = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}, \qquad c^* = (k^*)^\alpha - \delta k^*.$$

We restrict attention to interior paths with $c(t) > 0$ and $k(t) \geq 0$, so that $\log c(t)$ and $k(t)^\alpha$ are well defined. The Hamiltonian is

$$H(k, c, \lambda) \; = \; \log c - \log c^* + \lambda \left(k^\alpha - \delta k - c\right).$$

The Maximum Principle gives

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda = 0, \qquad \lambda' = -\lambda\left(\alpha k^{\alpha-1} - \delta\right), \qquad k' = k^\alpha - \delta k - c.$$

Substituting $\lambda = 1/c$ into the costate equation turns the canonical system into the planar system

$$\begin{aligned} k' &= k^\alpha - \delta k - c, \\ c' &= c\left(\alpha k^{\alpha-1} - \delta\right). \end{aligned}$$

At the steady state $\alpha(k^*)^{\alpha-1} = \delta$, so $(\lambda^*)' = 0$ and $\lambda^* = 1/c^* > 0$.

Before invoking Theorem 6.4.2 we must check that the objective converges along the candidate trajectory; this is the role of the constant $-\log c^*$ subtracted in the integrand.

The Jacobian of the planar system at (k^*, c^*) has both diagonal entries equal to $\alpha(k^*)^{\alpha-1} - \delta = 0$ and reduces to

$$\begin{bmatrix} 0 & -1 \\ -\mu^2 & 0 \end{bmatrix}, \qquad \mu = \sqrt{\alpha(1-\alpha)\, c^*(k^*)^{\alpha-2}} > 0,$$

[26] The steady-state pair (k^*, c^*) is obtained by imposing $\lambda' = 0$ and $k' = 0$ on the canonical system. Since $\lambda = 1/c > 0$, $\lambda' = 0$ forces $\alpha(k^*)^{\alpha-1} = \delta$, hence $k^* = (\alpha/\delta)^{1/(1-\alpha)}$; then $k' = 0$ yields $c^* = (k^*)^\alpha - \delta k^*$.

with eigenvalues $\pm\mu$. Two real eigenvalues of opposite sign mean that the steady state is a saddle: the eigenvalue $-\mu$ pulls nearby trajectories in, while $+\mu$ pushes them out. Geometrically, only a one-dimensional curve of initial conditions, the stable manifold (or saddle path), leads into the steady state; this is a local property, valid near (k^*, c^*). Off the saddle path a trajectory eventually leaves this neighborhood along the unstable direction: either consumption is driven to zero, or the economy overaccumulates and capital is pushed back down by depreciation, since $k^\alpha - \delta k - c < 0$ for k large. The global phase diagram confirms that no such path can be optimal, so the optimal trajectory issued from $k(0) = 1$ is the one lying on the saddle path, and along it $\lim_{t\to\infty}(k(t), c(t)) = (k^*, c^*)$.

Convergence along the saddle path is exponential. Near the steady state the nonlinear system behaves like its linearization, whose stable direction contracts at the rate μ fixed by the eigenvalue $-\mu$, so the linearized deviation decays like $e^{-\mu t}$. The true system differs from its linearization by terms that are quadratic in the deviation from the steady state, and these can slow the decay slightly; one can therefore guarantee a rate arbitrarily close to μ, but not μ itself. Precisely, for every $\tilde{\mu} \in]0, \mu[$ there is a constant $C > 0$ with $|c(t) - c^*| \leq C\, e^{-\tilde{\mu} t},\ t \geq 0$.[27] This exponential decay is exactly what makes the objective converge. Since $c(t) \to c^* > 0$, there exists T with $c(t) \geq c^*/2$ for $t \geq T$, and the mean value

[27] This is a standard refinement of the Stable Manifold Theorem (Theorem 4.2.1): on the stable manifold the linearized dynamics contract at rate μ, and the quadratic nonlinear remainder only degrades the guaranteed rate to any $\tilde{\mu} < \mu$. See Perko (2013), Section 2.7.

theorem applied to the logarithm gives, for $t \geq T$,

$$|\log c(t) - \log c^*| \leq \frac{2}{c^*}\,|c(t) - c^*| \leq \frac{2C}{c^*}\,e^{-\tilde{\mu}t}.$$

The integrand is therefore dominated by an integrable exponential on $[T, \infty[$, while on $[0, T]$ it is continuous and hence bounded, so the objective converges absolutely along the candidate trajectory.

Had the integrand been $\log c(t)$ alone, it would converge to $\log c^* \neq 0$ in general and the improper integral would diverge: the constant $-\log c^*$ is exactly the subtraction that makes the integrand vanish at the steady state. Along this trajectory $\lim\limits_{t\to\infty} \lambda^*(t) = 1/c^* \neq 0$, while

$$\lim_{t\to\infty} H(k^*, c^*, \lambda^*) = \log c^* - \log c^* + \frac{1}{c^*}\,((k^*)^\alpha - \delta k^* - c^*) = 0,$$

the form predicted by Theorem 6.4.2.

$\diamond\diamond\diamond$

THE CURRENT-VALUE HAMILTONIAN

Almost every model below is posed in discounted form: an exponential weight $e^{-\rho t}$ multiplies a payoff that is otherwise independent of time. This sub-class warrants its own notation, for two reasons. First, a change of variable removes the discount factor from the optimality conditions, making them autonomous whenever f is, which simplifies the phase-diagram analysis. Second, here the transversality condition can be strengthened to the product form $\lim\limits_{t\to\infty} e^{-\rho t}\psi^*(t)\,x^*(t) = 0$, which is precisely the one used in every application of this section. The discounted problem $\mathcal{P}^{\infty}_{c,\rho}$ is

$$\begin{aligned}
&\max_{u\in\mathcal{U}} && J = \int_{t_0}^{\infty} e^{-\rho t} F(x(t), u(t))\, dt \\
&\text{s. t. :} && x'(t) = f(x(t), u(t), t), \\
& && x(t_0) = x_0, \\
& && \lim_{t\to\infty} \beta(t)\, x(t) \geq x_1,
\end{aligned} \tag{6.4.9}$$

with $\rho > 0$ and F independent of t. The present-value Hamiltonian is $H(x, u, \lambda, t) = e^{-\rho t} F(x, u) + \lambda\, f(x, u, t)$. Setting

$$\psi(t) = e^{\rho t} \lambda(t),$$

we define the current-value Hamiltonian,[28]

$$\mathcal{H}(x, u, \psi, t) \;=\; F(x, u) + \psi\, f(x, u, t), \tag{6.4.10}$$

in terms of which, for an interior optimal control u^*, the canonical equations become

$$\frac{\partial \mathcal{H}}{\partial u}(x^*, u^*, \psi^*, t) = 0, \tag{6.4.11}$$

$$(\psi^*)'(t) = \rho\, \psi^*(t) - \frac{\partial \mathcal{H}}{\partial x}(x^*, u^*, \psi^*, t), \tag{6.4.12}$$

$$(x^*)'(t) = f(x^*(t), u^*(t), t),$$

with $x^*(t_0) = x_0$ and $\lim_{t\to\infty} \beta(t)\, x^*(t) \geq x_1$. The exponential weight appearing here deserves a name of its own, since it enters both the objective of (6.4.9) and, as noted earlier, the terminal condition through the weight β.

[28] The transformation $\psi = e^{\rho t}\lambda$ removes the discount factor from the optimality conditions and is purely a change of variable: it does not alter the set of optimal trajectories.

Definition 6.4.1. (Discount factor). The discount factor associated with an instantaneous discount rate $r : [t_0, \infty[\to]0, \infty[$ is $\exp\left\{-\int_{t_0}^{t} r(s)\, ds\right\}$. When r is constant, with $r(s) = \rho$ for all s, this reduces to $e^{-\rho(t-t_0)}$ and, with $t_0 = 0$, simply to $e^{-\rho t}$.

The next definition isolates the monotonicity property used in the theorem below.

Definition 6.4.2. (Weak monotonicity). A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is weakly monotone on a convex set $S \subseteq \mathbb{R}^n$ if, for each coordinate $i \in \{1, \ldots, n\}$, the partial function $x_i \mapsto \varphi(x_1, \ldots, x_i, \ldots, x_n)$ is monotone on S, either non-decreasing or non-increasing, with the direction possibly depending on i. Equivalently, where the gradient exists, each partial derivative $\partial\varphi/\partial x_i$ has constant sign on S, although different partials may differ in sign.

The following theorem is the workhorse of the applications. Its first part carries the Hamiltonian transversality condition over to the discounted setting; its second part strengthens it, under monotonicity and boundedness hypotheses, to the product form $\lim_{t\to\infty} e^{-\rho t}\psi^*(t)\, x^*(t) = 0$, which is both easier to verify in the models below and exactly the ingredient required by the sufficiency theorem that closes the section.

Theorem 6.4.3. (Transversality with discount factor). Consider the $\mathcal{P}^{\infty}_{c,\rho}$ problem under (A1)–(A3). Suppose the optimal interior trajectories (x^*, u^*, ψ^*) exist and that, for t large enough, the value function

$$J(x,t) \;=\; \max_{u\in\mathcal{U}_{t,x}} \int_t^{\infty} e^{-\rho s} F(x(s), u(s))\, ds$$

is of class C^1 in a neighborhood of the optimal trajectory and satisfies $\lim_{t\to\infty} \partial J(x^*(t),t)/\partial t = 0$. Then (6.4.11)–(6.4.12) hold along (x^*, u^*, ψ^*) together with the present-value transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\,\mathcal{H}(x^*(t), u^*(t), \psi^*(t), t) \;=\; 0. \tag{6.4.13}$$

Suppose, in addition, that there exist constants $m, M > 0$ with

$$\left|\frac{\partial f(x,u,t)}{\partial u}\right| \geq m, \qquad \left|\frac{\partial F(x,u)}{\partial u}\right| \leq M, \tag{6.4.14}$$

for all admissible (x, u, t), that F is weakly monotone in (x, u) and f is weakly monotone in (x, u, t), and that either $\lim_{t\to\infty} x^*(t)$ exists and is finite, or $\lim_{t\to\infty} (x^*)'(t)/x^*(t) = y \in \mathbb{R}$. Then the strengthened transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\,\psi^*(t)\,x^*(t) \;=\; 0 \tag{6.4.15}$$

holds.

The first part follows by applying Theorem 6.4.2 to the present-value Hamiltonian; the strengthened form (6.4.15) is technical and is proved in Appendix 6.A.3 (Proof of Theorem 6.4.3).

Remark 6.4.3. The two alternatives in the last hypothesis cover the two asymptotic regimes met in the applications. The first, $\lim_{t\to\infty} x^*(t)$ finite, applies to models that converge to a steady state, such as the Ramsey–Cass–Koopmans model. The second, $\lim_{t\to\infty} (x^*)'(t)/x^*(t) = y$, applies to models whose optimal trajectories grow asymptotically at a constant proportional rate, as along the balanced growth paths of the AK and Lucas models; there the parameter restrictions that guarantee convergence of the objective also deliver (6.4.15).

Before putting these conditions to use, it is worth seeing why so much care was needed. The next example, due to Halkin Halkin (1974), exhibits an infinite-horizon problem and an optimal pair along which both seemingly natural limit conditions on the costate fail; only the Hamiltonian condition of Theorem 6.4.2 survives.

Examples

Example 6.4.2. (Halkin's counterexample). Consider the undiscounted problem

$$\begin{aligned} \max_{u\in\mathcal{U}} \qquad & J = \int_0^\infty u(t)\,(1-x(t))\,dt \\ \text{s. t. :}\quad & x'(t) = u(t)\,(1-x(t))\,, \\ & x(0) = 0, \\ & x(t) \text{ free at infinity,} \end{aligned}$$

with $\Omega = [0,1]$. Here $F = f = u(1-x)$. Integrating the state equation, $x(t) = 1 - \exp\left\{-\int_0^t u(\tau)\,d\tau\right\}$. Since $u(\tau) \geq 0$, the map $t \mapsto \int_0^t u(\tau)\,d\tau$ is non-decreasing, so it tends either to a finite limit or to ∞. In the latter case $\lim_{t\to\infty} x(t) = 1$ and the objective equals

$$\int_0^\infty u(t)\,(1-x(t))\,dt = \int_0^\infty x'(t)\,dt = \lim_{t\to\infty} x(t) - x(0) = 1;$$

in the former case $\lim_{t\to\infty} x(t) < 1$ and the objective is strictly smaller than 1. The optimal value is therefore 1, attained in particular by every constant control $u = u_0$ with $u_0 \in (0,1]$, for which $x^*(t) = 1 - e^{-u_0 t}$. The Hamiltonian is $H = u(1-x)(1+\lambda)$, and the costate equation $\lambda' = -\partial H/\partial x = u(1+\lambda)$, together with the maximum condition (note that $\partial H/\partial u = (1-x)(1+\lambda)$), admits the constant solution $\lambda^* = -1$. Then

$$\lim_{t\to\infty} \lambda^*(t) = -1 \neq 0, \qquad \lim_{t\to\infty} \lambda^*(t)\, x^*(t) = -1 \neq 0,$$

so neither $\lim_{t\to\infty} \lambda^*(t) = 0$ nor $\lim_{t\to\infty} \lambda^*(t)\, x^*(t) = 0$ holds at this optimum, showing that neither can be a necessary condition for optimality in the infinite-horizon case. The transversality condition that does hold is $\lim_{t\to\infty} H = 0$ (Theorem 6.4.2).

Example 6.4.3. Example 6.4.1 provides another illustration that $\lim_{t\to\infty} \lambda^*(t)\, x^*(t) = 0$ may fail without discounting: writing $k^*(t)$ for the optimal trajectory and k^* for its limit,

$$\lim_{t\to\infty} \lambda^*(t)\, k^*(t) \;=\; \frac{k^*}{c^*} \;>\; 0.$$

◇◇◇

SUFFICIENT CONDITIONS

The Maximum Principle identifies candidate trajectories but does not establish their optimality. The gap is closed by the infinite-horizon counterpart of the Mangasarian–Arrow sufficiency theorem of the finite-horizon case (Section 6.2); the key requirement is concavity of the maximized Hamiltonian in the state. Each hypothesis plays a definite role: concavity gives a pointwise comparison between the candidate and any competing admissible trajectory, while the strengthened transversality condition (6.4.15) and the feasibility condition (6.4.16) together fix the sign of the boundary term at infinity.

Theorem 6.4.4. (Sufficient conditions: discounted infinite horizon). For the $\mathcal{P}^{\infty}_{c,\rho}$ problem in (6.4.9) under (A1)–(A3), with $\mathcal{H}$ as in (6.4.10), let (x^*, u^*, ψ^*) satisfy the current-value optimality conditions (6.4.11)–(6.4.12), the state equation, the maximum condition

$$\mathcal{H}(x^*(t), u^*(t), \psi^*(t), t) = \max_{v \in \Omega} \mathcal{H}(x^*(t), v, \psi^*(t), t),$$

and the strengthened transversality condition (6.4.15). Assume further that $J(x^*(t), t)$ is finite for all $t \geq t_0$ and that, for every admissible state trajectory x,

$$\lim_{t \to \infty} e^{-\rho t}\, \psi^*(t)\, x(t) \;\geq\; 0. \tag{6.4.16}$$

Set $M(x, \psi, t) = \max_{v \in \Omega} \mathcal{H}(x, v, \psi, t)$. If $x \mapsto M(x, \psi, t)$ is concave for each (ψ, t), then (x^*, u^*) is a global maximizer of $\mathcal{P}^{\infty}_{c,\rho}$. If M is strictly concave in x, then x^* is unique; uniqueness of u^* requires, in addition, strict concavity of $\mathcal{H}$ in u.

The proof parallels the finite-horizon Mangasarian–Arrow argument of Section 6.2 and is given in Appendix 6.A.3 (Proof of Theorem 6.4.4). In practice the theorem is applied as follows: solve the canonical system from the Maximum Principle, verify the strengthened transversality condition through Theorem 6.4.3, and check concavity of the maximized Hamiltonian in the state; the candidate path is then the optimum. Finally, the applications repeatedly differentiate the first-order condition in time, which requires the optimal control to be free of jumps. Under the additional strict-concavity and compactness hypotheses stated in Appendix 6.A.3

(Theorem 6.A.4), the optimal control $u^* : [t_0, \infty[\to \Omega$ is continuous, which legitimizes that step.

Theorem 6.4.4 and the continuity result conclude the theoretical development of this section. Two further questions can be pursued in the references below: the existence and uniqueness of solutions to the $\mathcal{P}_c^\infty$ problem, and the concavity and differentiability of the value function J. For existence of solutions we refer to Cesari (1966) and Baum (1976), for differentiability of the value function to Benveniste and Scheinkman (1979), for its duality and concavity properties to Benveniste and Scheinkman (1982), and for the Maximum Principle and transversality condition in concave infinite-horizon models to Araujo and Scheinkman (1983).

We now turn to the application of these results to growth and dynamic-macroeconomic models, where the $\mathcal{P}_c^\infty$ problem arises naturally. These models share a common structure, differing mainly in the terminal-state condition: the state at infinity is sometimes left free and sometimes constrained by $\lim\limits_{t\to\infty} \beta(t)\, x(t) \geq x_1$. We begin by revisiting the Ramsey–Cass–Koopmans model, first in its baseline form and then with taxation, and we then develop the AK growth model, the Lucas human-capital model, the Romer model of innovation-driven growth with a continuum of intermediate products, Hotelling's rule for exhaustible resources, the Clark fishery, the Nerlove–Arrow advertising model, and Tobin's Q. We use the tools developed in Chapter 4 to study the dynamical systems that arise from the necessary conditions for optimality.

Examples

Example 6.4.4. (RCK model revisited). Recall the $\mathcal{P}^{\infty}_{c,\rho}$ problem obtained in (6.0.7):

$$\begin{aligned}
\max_{c(t)\,\in\,\Omega(k(t))} \qquad & U = \int_0^{\infty} e^{-\beta\, t}\, u(c(t))\, dt \\
\text{s. t. :} \qquad & k'(t) = f(k(t)) - (n+\delta)\, k(t) - c(t), \\
& k(0) = k_0, \quad 0 < k_0 < \bar{k}, \\
& \Omega(k(t)) = [\, 0, f(k(t))\,]\,.
\end{aligned}$$

Recall that the admissible set $\Omega(k(t)) = [0,\ f(k(t))]$ depends on the state k, so this is a mixed state-control constraint $u(t) \in \Omega(x(t))$ rather than a pure control constraint $u(t) \in \Omega(t)$, the case treated in Section 6.1. The optimal path, however, is interior: $c = 0$ is excluded by the Inada condition $\lim_{c\to 0^+} u'(c) = \infty$, which would force $\psi = \infty$, and $c = f(k)$ is suboptimal, since it gives $k'(t) = -(n+\delta)\, k(t)$ and hence $k(t) \to 0$. The constraint is thus inactive along the optimum, and the unconstrained Maximum Principle applied below is justified.

The Hamiltonian is

$$H(k, c, \lambda, t) \;=\; e^{-\beta\, t}\, u(c) + \lambda \big(f(k) - (n+\delta)\, k - c\big).$$

Introducing the current-value costate variable $\psi = e^{\beta\, t}\, \lambda$, we obtain the current-value Hamiltonian:

$$\mathcal{H}(k, c, \psi, t) \;=\; u(c) + \psi \big(f(k) - (n+\delta)\, k - c\big). \tag{6.4.17}$$

The Maximum Principle yields

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial c} &= u'(c) - \psi \;=\; 0, \\
\frac{\partial^2 \mathcal{H}}{\partial c^2} &= u''(c) \;<\; 0, \\
\psi' &= \beta\,\psi - \frac{\partial \mathcal{H}}{\partial k} \;=\; \psi\,\big(\beta + n + \delta - f'(k)\big), \\
k' &= \frac{\partial \mathcal{H}}{\partial \psi} \;=\; f(k) - (n+\delta)\,k - c, \qquad k(0) = k_0 > 0,
\end{aligned}
$$

together with the transversality condition

$$\lim_{t\to\infty} e^{-\beta\,t}\,\psi(t)\,k(t) \;=\; 0. \tag{6.4.18}$$

The product form (6.4.18) is not delivered by Theorem 6.4.3: its boundedness hypothesis fails here, since $\partial F/\partial c = u'(c)$ is unbounded as $c \to 0^+$ under the Inada condition. We therefore read (6.4.18) as the relevant no-overaccumulation condition and verify it directly along the candidate path, where $\psi(t)$ and $k(t)$ remain bounded and $e^{-\beta t} \to 0$.

From the FOC $\psi = u'(c)$ together with the Inada condition $u'(c) > 0$ for $c > 0$, the multiplier satisfies $\psi(t) > 0$ along the optimal path. This positivity is consistent with the standard shadow-price interpretation of ψ (see Section 6.3): ψ represents the marginal value, in current units, of an additional unit of capital, and that marginal value is naturally non-negative whenever capital is productive.

The Hessian of $\mathcal{H}$ with respect to (k, c) is $H_{\mathcal{H}}(k, c) = \operatorname{diag}(\psi f''(k), u''(c))$, which is negative definite, since $\psi > 0$, $f''(k) < 0$ and $u''(c) < 0$. Hence $\mathcal{H}$ is strictly concave in

(k, c), and the maximized current-value Hamiltonian $M(k, \psi, t) = \max_{c \in \Omega(k)} \mathcal{H}(k, c, \psi, t)$ inherits strict concavity in k. Furthermore, the candidate costate satisfies $\psi^*(t) = u'(c^*(t)) > 0$, and every admissible state path obeys $0 \leq k(t) \leq \bar{k}$, so $\lim\limits_{t \to \infty} e^{-\beta t} \psi^*(t) k(t) \geq 0$. By the standard sufficiency theorem for discounted infinite-horizon problems (Theorem 6.4.4), the necessary conditions are also sufficient: the candidate path is the unique global optimum.

If $f'(k)$ admits an explicit antiderivative against the discount factor, ψ can be obtained in closed form; c is then recovered from $u'(c) = \psi$ and substituted into the state equation to obtain k. In general, however, this is not feasible, and we proceed qualitatively.

By Inada's conditions on u,

$$\lim_{c \to 0^+} u'(c) \;=\; \infty, \qquad \lim_{c \to \infty} u'(c) \;=\; 0, \tag{6.4.19}$$

together with the strict monotonicity and continuity of u', the equation $u'(c) = \psi$ admits, for every $\psi > 0$, a unique solution $c^* = c^*(\psi) > 0$ (Intermediate Value Theorem). Therefore

$$u'(c^*) \;=\; \psi. \tag{6.4.20}$$

Differentiating (6.4.20) with respect to t via the chain rule gives (omitting for simplicity the $*$)

$$\begin{aligned} u''(c)\, c' \;&=\; \psi' \\ &=\; \psi\,\big(\beta + n + \delta - f'(k)\big) \\ &=\; u'(c)\,\big(\beta + n + \delta - f'(k)\big), \end{aligned}$$

so that

$$c' \;=\; -\,\frac{u'(c)}{u''(c)}\,\big(f'(k) - (n + \delta + \beta)\big). \tag{6.4.21}$$

Combining (6.4.21) with the state equation, we obtain the planar autonomous system

$$\begin{aligned} k' &= f(k) - (n+\delta)\,k - c, && (6.4.22) \\ c' &= -\frac{u'(c)}{u''(c)}\big(f'(k) - (n+\delta+\beta)\big). && (6.4.23) \end{aligned}$$

Equivalently, defining the (Arrow–Pratt) coefficient of relative risk aversion [Mas-Colell et al. (1995)]

$$\theta(c) = -\,c\,\frac{u''(c)}{u'(c)},$$

equation (6.4.23) can be rewritten as the following equation

$$\frac{c'}{c} = \frac{1}{\theta(c)}\left(f'(k) - (n+\delta+\beta)\right), \qquad (6.4.24)$$

which is, in general, a nonlinear differential equation.

Phase-plane analysis.

The isoclines of (6.4.22)–(6.4.23) are

$$\begin{aligned} k' = 0 \iff c &= f(k) - (n+\delta)\,k, \\ c' = 0 \iff f'(k) &= n+\delta+\beta. \end{aligned}$$

The isocline $k' = 0$ is the curve $c(k) = f(k) - (n+\delta)\,k$, the difference between the production curve and the depreciation line (see Figure 6.9). It rises from zero up to a maximum at $k = \hat{k}$, where $f'(\hat{k}) = n + \delta$, and falls back to zero at $k = \bar{k}$. The locus $c' = 0$ is the vertical line $k = k^*$, with $f'(k^*) = n + \delta + \beta$. Since $\beta > 0$ and $f'' < 0$, we have $k^* < \hat{k}$.

These isoclines partition the phase plane into four regions, in which the vector field has a horizontal component (HD) and a vertical component (VD):

$$\text{HD}: \begin{cases} k' > 0 & \Leftrightarrow \quad c < f(k) - (n+\delta)\,k, \\ k' < 0 & \Leftrightarrow \quad c > f(k) - (n+\delta)\,k, \end{cases}$$

$$\text{VD}: \begin{cases} c' > 0 & \Leftrightarrow \quad f'(k) > n+\delta+\beta, \\ c' < 0 & \Leftrightarrow \quad f'(k) < n+\delta+\beta. \end{cases}$$

The vector field suggests that P^* is a saddle-point steady state. To establish this rigorously, we compute the Jacobian of (6.4.22)–(6.4.23) at $P^* = (k^*, c^*)$:

$$J(P^*) \;=\; \begin{bmatrix} f'(k^*) - (n+\delta) & -1 \\ -\dfrac{u'(c^*)}{u''(c^*)}\, f''(k^*) & 0 \end{bmatrix}.$$

Since $u' > 0$, $u'' < 0$ and $f'' < 0$,

$$\det J(P^*) \;=\; -\,\frac{u'(c^*)}{u''(c^*)}\, f''(k^*) \;<\; 0,$$

so the linearization at P^* has two real eigenvalues of opposite sign, and P^* is a hyperbolic saddle. By the H-G Theorem (Theorem 4.1.1), near P^* the nonlinear system behaves qualitatively like its linearization, so the saddle structure carries over. Its economic content is determinacy: k_0 is predetermined, while c_0 is free, and the saddle property says that, for k_0 near k^*, exactly one choice of c_0 places the economy on the local stable manifold converging to P^*. That this stable manifold extends to a single-valued saddle path selecting a unique c_0 for every admissible k_0 is a global property, read off the phase diagram below rather than from the local linearization. Figure 6.9 illustrates the vector field in each region.

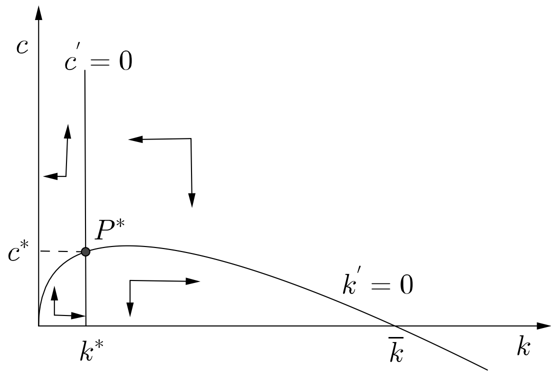

Figure 6.9. Vector field and phase-diagram regions for the RCK model.

By the Stable Manifold Theorem (Theorem 4.2.1), the system possesses a one-dimensional local stable manifold $W^s_{\text{loc}}(P^*)$ along which trajectories converge to P^* (Figure 6.10).

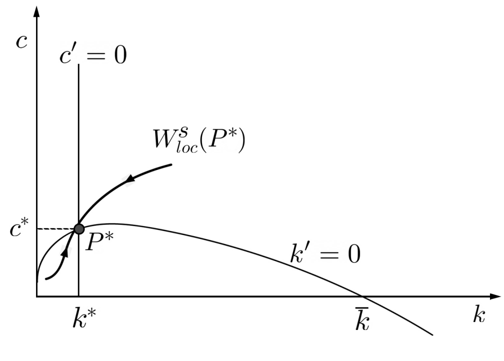

Figure 6.10. Stable manifold of the saddle steady state for the RCK model.

Conclusions of the model.

The phase diagram in Figure 6.11 displays representative trajectories of (6.4.22)–(6.4.23). Not all of them are economically meaningful. Indeed, if consumption is too high, per-capita capital collapses to a negligible level, which is undesirable for production; if consumption is too low, capital overaccumulates while consumption remains insignificant, which is also undesirable for households. In the long run, every trajectory will reach one of these two outcomes unless it lies on the stable manifold $W^s_{\mathrm{loc}}(P^*)$.

Every trajectory in $W^s_{\mathrm{loc}}(P^*)$ satisfies the transversality condition. Indeed, as t tends to infinity, $k(t)$ converges to k^* and $c(t)$ converges to c^*. Hence $u'(c(t))$ converges to $u'(c^*)$, a finite positive number, and therefore

$$\begin{aligned}\lim_{t\to\infty} e^{-\beta t}\,\psi(t)\,k(t) \;&=\; \lim_{t\to\infty} e^{-\beta t}\,u'(c(t))\,k(t)\\ &=\; u'(c^*)\,k^*\,\lim_{t\to\infty} e^{-\beta t} \;=\; 0.\end{aligned}$$

Since $k(0) = k_0$ is given, the optimal solution requires choosing the unique initial consumption c_0 such that the trajectory starting at (k_0, c_0) lies on $W^s_{\mathrm{loc}}(P^*)$.

Comparative statics.

In the model we fixed the productivity level at $A = 1$ and wrote the intensive production simply as $f(k)$; restoring it, output per worker is $Af(k)$, where $A > 0$ measures total factor productivity. Compare two levels $A_1 < A_2$. The marginal product of capital is then $Af'(k)$, so a higher A raises it proportionally at every k. Accordingly, the modified golden rule reads $Af'(k^*) = n + \delta + \beta$, that is, $f'(k^*) = (n + \delta + \beta)/A$. Since f' is strictly decreasing, dividing by the larger A_2 lowers the target value of f', so its solution satisfies $k_2^* > k_1^*$, and the corresponding $c^* = Af(k^*) - (n + \delta)\, k^*$ rises as well. Hence the steady state (k^*, c^*) shifts to the right and up. An economy with better technology therefore sustains both higher capital per worker and higher consumption per capita in the long run; the convergence speed, however, depends on the curvature of u and f near the steady state. The same comparative-static shift occurs when $n+\delta$ decreases, as is typically the case in developed economies, where capital depreciates slowly and per-capita population growth is small. For a modern textbook treatment, including extensions to technological progress, open economies, and fiscal policy, we refer the reader to Barro and i Martin (2003).

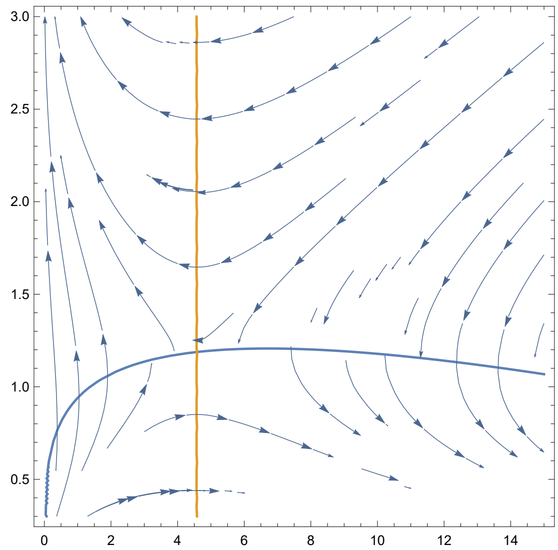

Figure 6.11. Phase diagram for the RCK model.

Example 6.4.5. (Numerical example). This example is taken from Shone (2012). Let

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \qquad f(k) = k^{0.25},$$

and consider the parameters $\beta = 0.02$, $n = 0.01$, $\delta = 0.05$, $\theta = 0.5$, with initial capital $k_0 = 2$. With $\theta = 0.5$, $u(c) = 2\sqrt{c} - 2$, so $u'(c) = c^{-1/2}$; the additive constant does not affect the optimization. Hence the discounted $\mathcal{P}^{\infty}_{c,\rho}$ problem of Example 6.4.4 reads

$$\begin{aligned}
\max_{c(t)\,\in\,\Omega(k(t))} \qquad & U = \int_0^{\infty} 2\,e^{-0.02\,t}\sqrt{c(t)}\,dt \\
\text{s. t. :} \qquad & k'(t) = k(t)^{0.25} - 0.06\,k(t) - c(t), \\
& k(0) = 2, \\
& \Omega(k(t)) = [\,0, f(k(t))\,]\,.
\end{aligned}$$

The current-value Hamiltonian is

$$\mathcal{H}(k, c, \psi, t) = 2\sqrt{c} + \psi\left(k^{0.25} - 0.06\,k - c\right).$$

The Maximum Principle gives

$$\begin{aligned}
\psi &= c^{-1/2}, \\
\psi' &= \psi\left(0.08 - 0.25\,k^{-0.75}\right), \\
k' &= k^{0.25} - 0.06\,k - c.
\end{aligned}$$

together with the transversality condition $\lim_{t\to\infty} e^{-0.02\,t}\,\psi(t)\,k(t) = 0$.

Differentiating $\psi = c^{-1/2}$ with respect to t and substituting yields the planar system

$$\begin{aligned}
k' &= k^{0.25} - 0.06\,k - c, \\
c' &= \left(0.5\,k^{-0.75} - 0.16\right)c, \qquad c > 0.
\end{aligned}$$

There is a unique steady state $(k^*, c^*) \approx (4.568,\, 1.188)$. The Jacobian

$$J(k^*, c^*) \approx \begin{bmatrix} 0.02 & -1 \\ -0.0312 & 0 \end{bmatrix}$$

has eigenvalues $\lambda_1 \approx -0.1669$ and $\lambda_2 \approx 0.1869$; therefore (k^*, c^*) is a saddle steady state. The associated stable and unstable subspaces are

$$\begin{aligned} \lambda_1 \approx -0.1669 : E^s &= \left\{ \alpha \begin{bmatrix} 1 \\ 0.1869 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \\ \lambda_2 \approx 0.1869 : E^u &= \left\{ \alpha \begin{bmatrix} 1 \\ -0.1669 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \end{aligned}$$

and the corresponding affine manifolds at (k^*, c^*) are

$$E^s\left((k^*, c^*)\right) = E^s + \begin{bmatrix} k^* \\ c^* \end{bmatrix}, \qquad E^u\left((k^*, c^*)\right) = E^u + \begin{bmatrix} k^* \\ c^* \end{bmatrix}.$$

By the Stable Manifold Theorem, $W^s_{\mathrm{loc}}(k^*, c^*)$ is tangent at (k^*, c^*) to the line

$$c = c^* + 0.1869\,(k - k^*) \approx 0.1869\,k + 0.3341.$$

Figure 6.12 shows the phase diagram of the model together with this stable affine manifold.

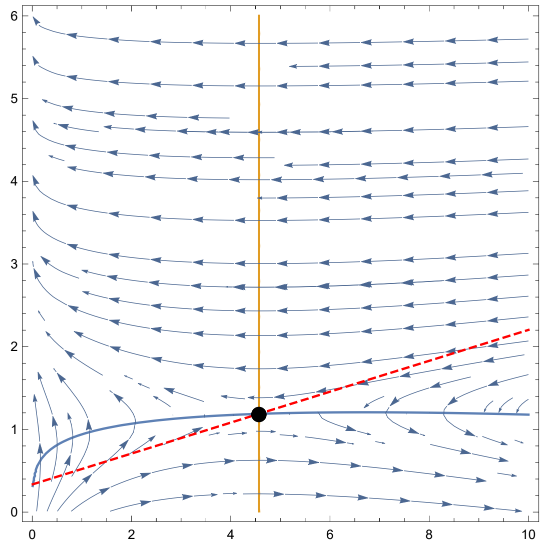

Figure 6.12. Phase diagram for the numerical RCK example.

Since this is the RCK model of Example 6.4.4 with explicit u and f, the verification carries over. Here $u''(c) = -\frac{1}{2}\,c^{-3/2} < 0$ and $f''(k) = -0.1875\,k^{-1.75} < 0$, so the current-value Hamiltonian is strictly concave in (k, c); the candidate costate satisfies $\psi^*(t) = c^*(t)^{-1/2} > 0$, and every admissible path obeys $0 \le k(t) \le \bar{k}$. By Theorem 6.4.4 the trajectory on $W^s_{\text{loc}}(k^*, c^*)$ is the unique global optimum. Along it $\lim_{t\to\infty} c(t) = c^*$, so $\lim_{t\to\infty} u'(c(t)) = (c^*)^{-1/2}$ is finite and

$$\lim_{t\to\infty} e^{-0.02\,t}\,\psi(t)\,k(t) \;=\; (c^*)^{-1/2}\,k^*\,\lim_{t\to\infty} e^{-0.02\,t} \;=\; 0,$$

confirming the transversality condition.

Economic interpretation.

Shone's calibration is a numerical instance of the RCK saddle. Since $k_0 = 2 < k^* \approx 4.568$, the economy starts to the left of the steady state and converges to $(k^*, c^*) \approx (4.568, 1.188)$ by accumulating capital while consumption rises. Determinacy is explicit: k_0 is predetermined and c_0 is free, and the unique c_0 placing (k_0, c_0) on $W^s_{\text{loc}}(k^*, c^*)$, near the steady state the line $c \approx 0.1869\, k + 0.3341$, selects the optimal path; any other choice drives the economy to capital collapse or over-accumulation.

Example 6.4.6. (AK growth model). The AK model, due to Rebelo (1991), replaces the diminishing returns of the neoclassical production function by the linear technology

$$Y(t) \;=\; A\,K(t), \qquad A > 0,$$

where K measures broad capital (physical and human). With population growth n, depreciation δ, and CRRA preferences, the planner solves

$$\begin{aligned}
\max_{c(t)\,\geq\,0} \quad & U = \int_0^\infty e^{-(\rho-n)\,t}\,\frac{c(t)^{1-\theta}-1}{1-\theta}\,dt \\
\text{s. t. :} \quad & k'(t) = (A-\delta-n)\,k(t) - c(t), \\
& k(0) = k_0 > 0.
\end{aligned}$$

The current-value Hamiltonian is

$$\mathcal{H}(k, c, \psi, t) \;=\; \frac{c^{1-\theta}-1}{1-\theta} + \psi\;[\,(A-\delta-n)\,k - c\,]\,,$$

and the Maximum Principle yields

$$\begin{aligned}
c^{-\theta} \;&=\; \psi, \\
\psi' \;&=\; (\rho+\delta-A)\,\psi, \\
k' \;&=\; (A-\delta-n)\,k - c,
\end{aligned}$$

together with the transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)\,t}\,\psi(t)\,k(t) \;=\; 0.$$

Differentiating the first condition with respect to time and substituting the costate equation yields

$$\frac{c'(t)}{c(t)} \;=\; \frac{A-\delta-\rho}{\theta} \;=\; g. \tag{6.4.25}$$

Phase diagram.

The isoclines are degenerate: $k' = 0$ gives $c = (A-\delta-n)\,k$, while $c'/c = 0$ collapses to the parameter restriction $A = \delta+\rho$, not a curve in the (k, c) plane.

This is a degenerate case in which the techniques used for the RCK model no longer apply. In the RCK model the two isoclines meet at an isolated, hyperbolic steady state, and the optimum is the stable manifold of a saddle, recovered by a phase-plane analysis. Here the consumption isocline degenerates into a parameter restriction rather than into a curve in (k, c), so there is no interior steady state to converge to and no saddle path to select. Under $A > \delta+\rho$ the economy instead grows without bound along a balanced growth path (BGP), a trajectory on which c, k, and Y grow at a single constant rate g, leaving their ratios constant. The analysis therefore proceeds differently: we first remove the common trend, which renders the BGP stationary, and we then select the optimal path by checking the transversality condition directly.

Setting $\tilde{c}(t) \;=\; c(t)\,e^{-g\,t}$ and $\tilde{k}(t) \;=\; k(t)\,e^{-g\,t}$, the system becomes

$$\tilde{k}' \;=\; (A-\delta-n-g)\,\tilde{k}-\tilde{c}, \qquad \tilde{c}' \;=\; 0.$$

The steady states of the detrended system are not isolated: since $\tilde{c} = (A - \delta - n - g)\,\tilde{k}$, so the steady states fill the balanced-growth ray, which is economically admissible (with $\tilde{c} > 0$) precisely when $A - \delta - n - g > 0$, equivalently $(1 - \theta)\,g < \rho - n$, the condition established below. What the optimum selects is the unique consumption-to-capital ratio $\tilde{c}_0/\tilde{k}_0 = A-\delta-n-g$ consistent with transversality: at the predetermined $\tilde{k}_0$ the economy starts on the ray and stays there, with c, k, and Y all growing at rate g and no transitional dynamics (Figure 6.13).

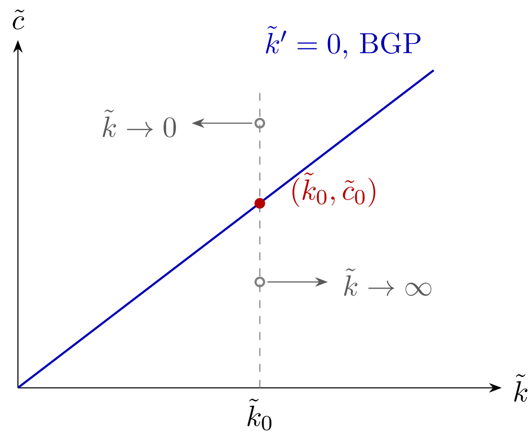

Figure 6.13. Only the choice $\tilde{c}_0$ on the BGP ray keeps $\tilde{k}$ constant; higher (lower) choices drive $\tilde{k}$ to 0 (∞).

The candidate path $c^*(t) = c_0\,e^{g\,t}$, $k^*(t) = k_0\,e^{g\,t}$, $\psi^*(t) = \psi(0)\,e^{(\rho+\delta-A)\,t}$ satisfies the necessary conditions of the Maximum Principle. We do not invoke Theorem 6.4.3 directly: under CRRA preferences, $|\partial F/\partial c| = u'(c) = c^{-\theta}$ is unbounded as c tends to 0 from the right. We therefore check the strengthened transversality condition by direct substitution. Multiplying the three exponentials,

$$e^{-(\rho-n)\,t}\,\psi^*(t)\,k^*(t) \;=\; \psi(0)\,k_0\,e^{(g+\delta+n-A)\,t},$$

so the exponent equals

$$-(A-\delta-n-g) \;=\; (1-\theta)\,g-(\rho-n),$$

where the second form uses $\theta\,g = A-\delta-\rho$. The limit is zero if and only if this exponent is negative, that is,

$$(1-\theta)\,g \;<\; \rho-n. \tag{6.4.26}$$

This condition, known as Lucas condition, controls the growing term $c^{1-\theta}/(1-\theta)$; the additive constant $-1/(1-\theta)$ contributes a multiple of

$$\int_0^{\infty} e^{-(\rho-n)\,t}\,dt,$$

which converges if and only if $\rho > n$. Under both $(1-\theta)\,g < \rho-n$ and $\rho > n$ the discounted utility integral converges, so the planner's problem is well-posed.

For sufficiency, Theorem 6.4.4 applies on the interior domain $c > 0$, where F and f are C^1 (the boundary $c = 0$, at which the CRRA payoff fails to be C^1 e for $\theta \geq 1$, is excluded along the optimum since $c^{-\theta}$ diverges). Condition (6.4.16) holds because $\psi^* > 0$ and $k(t) \geq 0$ on any admissible path, and the maximized current-value Hamiltonian

$$M(k,\psi,t) \;=\; \frac{\theta}{1-\theta}\,\psi^{(\theta-1)/\theta} - \frac{1}{1-\theta} + \psi\,(A-\delta-n)\,k$$

is linear in k. The candidate is therefore a global maximizer whenever (6.4.26) and $\rho > n$ both hold.

Economic interpretation.

The condition $A > \delta + \rho$ is what turns this model into a growth model: the marginal product of broad capital, A, must exceed its user cost $\delta + \rho$, the sum of the depreciation rate and the rate of time preference at which the planner discounts future utility. When this happens, (6.4.25) delivers a positive constant per-capita growth rate that is fully endogenous: it depends on preferences (ρ, θ) and on the technology level A, with no role for an exogenous progress rate. Patience (small ρ), high productivity (large A), and a high elasticity of intertemporal substitution (small θ) all raise g. The Lucas condition keeps the model well-posed: the effective discount rate must be high enough relative to the growth rate of utility, that is, the planner must not be too patient; otherwise the discounted-utility integral diverges and the optimum is undefined ($U = \infty$).

The linearity $Y = A\,K$ removes diminishing returns to broad capital, so the consumption-to-capital ratio is constant from $t = 0$ onward and, in logarithms, $\log k(t) = \log k_0 + g\,t$ and $\log c(t) = \log c_0 + g\,t$ are straight lines of common slope g (Figure 6.14). Because that slope is the same for every economy, the vertical distance between the log paths of two economies stays fixed at its initial value: a country that starts with a lower k_0 grows at exactly the same rate and never closes the gap. This is the crux of non-convergence. In the neoclassical model, diminishing returns give a poorer economy a higher marginal product, and hence faster growth, so it catches up; under the constant marginal product A that force is absent, and the growth rate g in (6.4.25) does not depend on the level of capital. Two economies with identical fundamentals but different k_0 therefore converge in growth rates but not in levels. This prediction motivated the empirical literature on conditional convergence and, more broadly, the endogenous-growth research program introduced in the next examples.

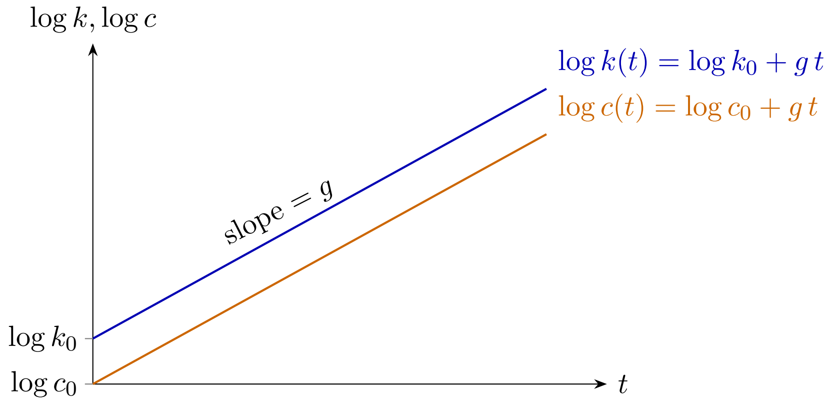

Figure 6.14. Log paths of k and c: parallel lines of common slope g. Since g is independent of k_0, initial gaps never close.

Example 6.4.7. (Lucas human capital model). In this example, we follow the analysis in the article "On the Mechanics of Economic Development", published in 1988 in the *Journal of Monetary Economics* [Lucas (1988)]. Output combines physical capital per worker k and human capital per worker h, with $u(t) \in [0, 1]$ the fraction of time devoted to work (so $1 - u(t)$ is spent in schooling) and population normalized to $L = 1$ and constant,

$$y(t) \;=\; k(t)^{\alpha}\,\left(u(t)\,h(t)\right)^{1-\alpha}, \qquad 0 < \alpha < 1.$$

Human capital accumulates in proportion to schooling time,

$$h'(t) \;=\; \xi\,\left(1 - u(t)\right) h(t), \qquad \xi > 0,$$

where ξ is the productivity of the schooling technology. With CRRA preferences and discount rate $\rho > 0$, the planner solves

$$\begin{aligned}
\max_{c \geq 0,\, u \in [0,1]} \qquad & U = \int_0^\infty e^{-\rho t}\, \frac{c(t)^{1-\theta} - 1}{1-\theta}\, dt \\
\text{s. t. :} \qquad & k'(t) = k^\alpha\, (u\, h)^{1-\alpha} - c - \delta\, k, \\
& h'(t) = \xi\, (1-u)\, h, \\
& k(0) = k_0, \quad h(0) = h_0.
\end{aligned}$$

The states are (k, h) and the controls are (c, u). With costates (ψ, μ) the current-value Hamiltonian is

$$\mathcal{H} \;=\; \frac{c^{1-\theta} - 1}{1-\theta} + \psi\, \left[k^\alpha\, (u\, h)^{1-\alpha} - c - \delta\, k \right] + \mu\, \xi\, (1-u)\, h.$$

We look for an interior optimum, meaning $u \in (0,1)$ and $c > 0$, so that the household both works and studies. This yields

$$\begin{aligned}
c^{-\theta} \;&=\; \psi, \\
(1-\alpha)\, \psi\, k^\alpha\, u^{-\alpha}\, h^{1-\alpha} \;&=\; \mu\, \xi\, h, \\
\psi' \;&=\; (\rho + \delta)\, \psi - \alpha\, \psi\, k^{\alpha-1}\, (u\, h)^{1-\alpha}, \\
\mu' \;&=\; \rho\, \mu - \mu\, \xi\, (1-u) \\
&\qquad - (1-\alpha)\, \psi\, k^\alpha\, u^{1-\alpha}\, h^{-\alpha},
\end{aligned}$$

together with the transversality conditions

$$\lim_{t \to \infty} e^{-\rho t}\, \psi(t)\, k(t) \;=\; 0, \qquad \lim_{t \to \infty} e^{-\rho t}\, \mu(t)\, h(t) \;=\; 0.$$

The economy has no steady state in levels. Along any admissible path $h > 0$, so the human-capital law $h' = \xi\, (1-u)\, h$ vanishes only at $u = 1$, that is, only if no time is spent in schooling. Indeed

$$h(t) \;=\; h_0\, \exp\left(\xi \int_0^t (1 - u(s))\, ds \right),$$

which is strictly increasing whenever $u < 1$. A level steady state would require $h' = 0$, hence $u = 1$, so no configuration with $k' = h' = c' = 0$ can be sustained while any time is devoted to schooling. Imposing $u = 1$, identically, removes the human-capital channel and collapses the model to a Ramsey economy with fixed h_0, which is not the case of interest. The relevant long-run object is therefore a balanced growth path rather than a stationary point.

Here a BGP is a trajectory along which the time allocation u is constant and the level variables (c, k, h, y) grow at a common constant rate g. Detrending the levels,

$$\tilde{k} \;=\; k\,e^{-g\,t}, \qquad \tilde{h} \;=\; h\,e^{-g\,t}, \qquad \tilde{c} \;=\; c\,e^{-g\,t},$$

each level grows at rate g on the BGP, so $\tilde{k}' = (k' - g\,k)\,e^{-g\,t} = 0$ and likewise $\tilde{h}' = \tilde{c}' = 0$: the BGP is a steady state of the detrended system, which can be analyzed with the usual tools for steady states and their stability.

We obtain g from the four FOCs. Differentiating $c^{-\theta} = \psi$,

$$\frac{c'}{c} \;=\; -\frac{1}{\theta}\,\frac{\psi'}{\psi}. \tag{6.4.27}$$

The costate equation for ψ gives $\psi'/\psi = (\rho + \delta) - \alpha\,k^{\alpha-1}\,(u\,h)^{1-\alpha}$; writing $r = \alpha\,k^{\alpha-1}\,(u\,h)^{1-\alpha}$ for the marginal product of capital,

$$\frac{c'}{c} \;=\; \frac{r - \rho - \delta}{\theta}, \tag{6.4.28}$$

so on the BGP, where $c'/c = g$, the return r is constant. The FOC for u, after canceling one factor of h, gives

$$\frac{\psi}{\mu} \;=\; \frac{\xi}{1-\alpha}\left(\frac{u\,h}{k}\right)^{\alpha}. \tag{6.4.29}$$

On the BGP u is constant and k, h grow at the same rate, so $u\,h/k$, and hence ψ/μ, is constant; differentiating its logarithm,

$$\frac{\psi'}{\psi} \;=\; \frac{\mu'}{\mu}. \tag{6.4.30}$$

The costate equation for μ is

$$\frac{\mu'}{\mu} \;=\; \rho - \xi\,(1-u) - (1-\alpha)\,\psi\,k^{\alpha}\,u^{1-\alpha}\,h^{-\alpha}/\mu.$$

The same FOC for u, divided by h and multiplied by u, identifies the last term as $\xi\,u$, so $\mu'/\mu = \rho-\xi$. By (6.4.30) also $\psi'/\psi = \rho-\xi$, and (6.4.27) gives

$$g \;=\; \frac{\xi-\rho}{\theta}. \tag{6.4.31}$$

Comparing (6.4.31) with (6.4.28) pins down the balanced return $r^* = \delta+\xi$.

For this candidate to be a genuine interior and admissible BGP, three things must hold: positive growth $g > 0$, an interior time allocation $u^* \in \,]0,1[$, and the two transversality conditions. From (6.4.31), $g > 0$ if and only if $\xi > \rho$. The time allocation follows from $h'/h = \xi\,(1-u^*) = g$, hence

$$u^* \;=\; 1-\frac{g}{\xi} \;=\; 1-\frac{\xi-\rho}{\theta\,\xi},$$

and $r^* = \alpha\,(u^*h/k)^{1-\alpha}$ together with (6.4.29) fixes the balanced ratio $z^* = k^*/h^*$. Now $u^* < 1$ if and only if $g > 0$, while $u^* > 0$ if and only if $g < \xi$. For the transversality conditions, along the BGP $\psi(t) = \psi(0)\,e^{(\rho-\xi)\,t}$ and $k(t) = k_0\,e^{g\,t}$ (and likewise μ, h); Theorem 6.4.3 does not apply, since $|\partial F/\partial c| = c^{-\theta}$ diverges as c approaches to 0. We therefore check directly

$$\lim_{t\to\infty} e^{-\rho\,t}\,\psi(t)\,k(t) \;=\; \lim_{t\to\infty} e^{(g-\xi)\,t} \;=\; 0 \quad\Longleftrightarrow\quad g<\xi,$$

and the condition for (μ, h) carries the same exponent $g - \xi$, hence the same inequality. Since $g < \xi$ is equivalent to

$$(1-\theta)\,\xi \;<\; \rho, \tag{6.4.32}$$

all three requirements reduce to the single chain $(1-\theta)\,\xi \;<\; \rho \;<\; \xi$. Within this range $g \in\,]0, \xi[$, $u^* \in\,]0, 1[$, and both transversality conditions hold, so the interior BGP exists and is admissible; (6.4.32) is the Lucas condition and also makes the discounted-utility integral converge.

For sufficiency, Theorem 6.4.4 applies: the running benefit and state equations are C^1; condition (6.4.16) holds because $\psi^*, \mu^* > 0$ and $k, h \geq 0$, and the maximized current-value Hamiltonian is concave in the states (k, h). For each fixed u,

$$\psi\, k^{\alpha}\,(u\, h)^{1-\alpha} - \psi\, \delta\, k + \mu\, \xi\, (1-u)\, h$$

is homogeneous of degree one in (k, h). Maximizing over $u \in [0, 1]$ gives a function that is linear in (k, h) on the interior region $u^* = C\, k/h < 1$, with $C = [\psi\,(1-\alpha)/(\mu\, \xi)]^{1/\alpha}$, and equals $\psi\, k^{\alpha}\, h^{1-\alpha} - \psi\, \delta\, k$ on the boundary region $u^* = 1$, where the schooling term vanishes; the two pieces meet with a continuous gradient at the seam $k = h/C$, so the maximized Hamiltonian is concave in (k, h). The BGP is therefore the global optimum whenever (6.4.32) holds.

Economic interpretation.

Long-run growth $g = (\xi - \rho)/\theta$ depends on the schooling productivity ξ, not on the level of physical capital, so the engine of sustained growth is human-capital accumulation, tempered by patience ρ and intertemporal substitution θ. It is the human-capital analogue of the AK growth rate (6.4.25), with ξ in the role of A. Read this way, cross-country income gaps reflect differences in how effectively economies turn time into human capital, through the quality of schooling and the returns to education, rather than differences in physical investment alone, which places education policy at the center of the long-run determinants of prosperity; see Barro and i Martin (2003).

Example 6.4.8. (Hotelling rule for exhaustible resources). A non-renewable resource of initial stock $S(0) = S_0 > 0$ is extracted at rate $q(t) \geq 0$ and sold in a competitive market with a continuous and strictly decreasing inverse demand $p(q)$. Let

$$U(q) \;=\; \int_0^q p(s)\,ds$$

be the gross surplus, so that $U'(q) = p(q)$. With constant unit extraction cost $c \geq 0$ and discount rate $r > 0$, the competitive (equivalently, efficient) extraction path maximizes the present value of net surplus over an infinite horizon:

$$\begin{aligned}
&\max_{q(t)\,\geq\,0} \quad \int_0^\infty e^{-r\,t}\,\left[\,U(q(t)) - c\,q(t)\,\right]dt\\
&\text{s.\,t.}: \quad S'(t) = -\,q(t),\\
&\qquad\quad\;\; S(0) = S_0, \quad S(t) \geq 0.
\end{aligned}$$

The state is S and the control is q. The current-value Hamiltonian is given by

$$\mathcal{H}(S, q, \psi, t) \;=\; U(q) - c\,q - \psi\,q.$$

The Maximum Principle gives

$$\frac{\partial \mathcal{H}}{\partial q} \;=\; p(q) - c - \psi \;=\; 0, \qquad \psi' \;=\; r\,\psi,$$

using $U'(q) = p(q)$. The costate law $\psi' = r\,\psi$ holds because $\mathcal{H}$ does not depend on S. Integrating it,

$$\psi(t) \;=\; \psi(0)\, e^{r\,t}, \tag{6.4.33}$$

so by the FOC the price net of cost grows at the rate of interest. Writing $p(t) = p(q(t))$ for the price along the path, $p(t) - c = \psi(t)$ gives $p'(t) = \psi'(t) = r\,\psi(t) = r\;[\,p(t) - c\,]$.

With linear demand $p(q) = a - b\,q$, the FOC $a - b\,q - c = \psi$ gives

$$q(t) \;=\; \frac{a - c - \psi(0)\, e^{r\,t}}{b}.$$

Extraction stays non-negative until $\psi(0)\, e^{r\,t}$ reaches the choke level $a - c$, that is, until

$$T^* \;=\; r^{-1}\, \ln\left(\frac{a-c}{\psi(0)}\right),$$

after which $q(t) = 0$. Integrating $S' = -q$ on $[0, T^*]$ with $S(T^*) = 0$ fixes $\psi(0)$ and hence the whole trajectory; exhaustion of a positive stock forces $\psi(0) < a - c$, so extraction is initially positive. On $[0, T^*[$ the constraint $S \geq 0$ is slack, and for $t \geq T^*$ the path $q = 0$, $S = 0$ remains feasible, so the state constraint never binds before exhaustion. Because linear demand has a finite choke price a, that ceiling is reached in finite time, so the resource is exhausted at the finite date T^*. Asymptotic depletion instead requires a demand with no choke price, so that extraction stays profitable however small the remaining stock.

Constant unit-elasticity demand $p(q) = B/q$ with $c = 0$ is the leading such case. Recall that the price elasticity of demand,

$\epsilon = (dq/dp)\,(p/q)$, measures the percentage change in quantity demanded per one-percent change in price, and a demand is unit-elastic when $|\epsilon| = 1$ at every price. For $p(q) = B/q$, equivalently $q = B/p$, one has $dq/dp = -B/p^2$, so $\epsilon = -1$ throughout, which names this case. Since

$$\int_0^q (B/s)\,ds$$

diverges, the gross surplus is here measured from a fixed reference level $\bar{q} > 0$,

$$U(q) \;=\; \int_{\bar{q}}^{q} p(s)\,ds \;=\; B\ln\left(\frac{q}{\bar{q}}\right),$$

which leaves $U'(q) = p(q)$ unchanged; the additive constant does not affect the optimization.

The FOC is $p(q) = \psi$, so $p(q(t)) = \psi(0)\,e^{r\,t}$; inverting the demand, $q(t) = (B/\psi(0))\,e^{-r\,t}$. The constant $\psi(0)$ is fixed by exhausting the stock:

$$\int_0^\infty q(t)\,dt = B/(r\,\psi(0)) = S_0,$$

so $B/\psi(0) = r\,S_0 = q(0)$. Hence $q(t) = r\,S_0\,e^{-r\,t}$, and integrating $S' = -q$,

$$S(t) \;=\; S_0 - \int_0^t r\,S_0\,e^{-r\,s}\,ds \;=\; S_0\,e^{-r\,t},$$

so $q(t) = r\,S(t)$ and $\lim_{t\to\infty} S(t) = 0$; the constraint $S \geq 0$ never binds. Since $\mathcal{H}$ is independent of S, the shadow price grows at rate r with no feedback from the stock, while q shrinks and carries S to zero. The three time paths are shown in Figure 6.15.

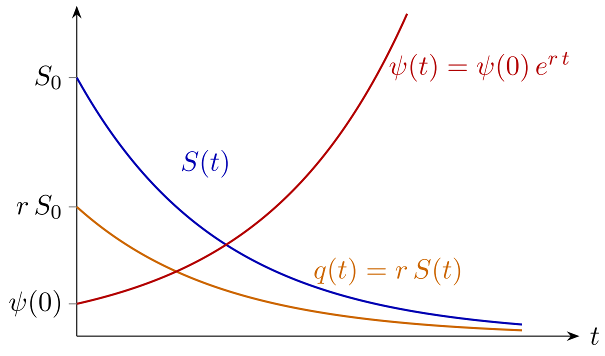

Figure 6.15. Stock, extraction, and shadow price in the unit-elasticity case ($p(q) = B/q$, $c = 0$): $q(t) = r\,S(t)$.

Here $F(S, q) = U(q) - c\,q$ does not depend on S, so $\partial F/\partial S = 0$. In the unit-elasticity case with $c = 0$, $\partial F/\partial q = p(q) = B/q > 0$ for all $q > 0$, so F is weakly monotone in q; the obstruction to applying Theorem 6.4.3 directly is instead that this derivative is unbounded as q converges to zero. We therefore verify the condition by hand. Since $e^{-r\,t}\,\psi^*(t) = \psi(0)$ is constant along the optimum and $S^*(t)$ converges to 0,

$$\lim_{t\to\infty} e^{-r\,t}\,\psi^*(t)\,S^*(t) \;=\; \psi(0)\,\lim_{t\to\infty} S^*(t) \;=\; 0.$$

For sufficiency, Theorem 6.4.4 applies: F and f are C^1, condition (6.4.16) holds because $\psi^* > 0$ and $S(t) \geq 0$, and the maximized Hamiltonian $M(S, \psi, t) = \max_{q\geq 0}\,\{U(q) - c\,q - \psi\,q\}$ is independent of S, hence concave in S. The candidate path is therefore the global optimum.

Economic interpretation.

Equation (6.4.33) is the Hotelling rule: the shadow value of an exhaustible resource grows at the rate of interest. It is a no-arbitrage condition, a notion borrowed from the theory of asset pricing in finance: an arbitrage is a riskless profit obtained from offsetting trades that require no net investment, and a no-arbitrage condition is the requirement that no such free profit be available, so that two ways of holding wealth with the same risk must offer the same return (the law of one price). Here the two ways of holding wealth are to extract one unit now and invest the proceeds at rate r, or to leave it in the ground and sell it later at an appreciated price. If ψ grew slower than r, extracting now and investing would dominate; if faster, waiting would dominate; the owner is indifferent between the two strategies only when $\psi'/\psi = r$ (6.4.33).

Example 6.4.9. (Clark model: optimal harvesting of a renewable resource). A fishery contains a stock $x(t)$ that grows logistically when unexploited. With harvest rate $h(t) \geq 0$,

$$x'(t) \;=\; r\,x(t)\left(1 - \frac{x(t)}{K_0}\right) - h(t), \quad x(0) = x_0 > 0,$$

where $r > 0$ is the intrinsic growth rate and $K_0 > 0$ the carrying capacity. Harvest sells at constant price p, with stock-dependent unit cost $c(x)$ strictly decreasing (denser stocks are cheaper to harvest). The planner discounts at rate $\rho > 0$ and solves

$$\begin{aligned} \max_{h(t)\geq 0} \qquad & \Pi = \int_0^\infty e^{-\rho t}\,[\,p - c(x)\,]\,h(t)\,dt \\ \text{s. t. :} \qquad & x' = r\,x\,(1 - x/K_0) - h, \\ & x(0) = x_0, \quad x(t) \geq 0. \end{aligned}$$

The state is x and the control is h. With costate ψ, the current-value Hamiltonian is

$$\mathcal{H}(x, h, \psi, t) \;=\; [\,p - c(x)\,]\,h + \psi\,[\,r\,x\,(1 - x/K_0) - h\,].$$

The Hamiltonian is linear in h, so the maximization over the control depends only on the sign of its coefficient $p - c(x) - \psi$. Where this coefficient is nonzero, the optimal control is bang-bang: since a function linear in the control is maximized at the boundary of the admissible set, the optimal policy takes only its extreme admissible values, switching abruptly between them.[29] The term bang-bang reflects the intuition that the control alternates forcefully between its two extremes, as in the on/off switching of a rocket thruster or a thermostat. Where, instead, the coefficient vanishes over a time interval, the maximization condition no longer pins down h; the control is then said to be singular, and is determined instead by the higher-order requirement that the coefficient remain zero along the entire interval, that is, $\psi = p - c(x)$; differentiating shows $\psi' = -c'(x)\, x'$, so ψ is constant only at the steady state, where x is constant. The interior steady state lies precisely in this singular regime, where

$$p - c(x) - \psi \;=\; 0, \tag{6.4.34}$$

and the costate equation is

$$\psi' \;=\; \rho\,\psi + c'(x)\,h - \psi\,r\,(1 - 2\,x/K_0).$$

At an interior steady state (x^*, h^*), we must have $h^* = r\,x^*\,(1 - x^*/K_0)$, and ψ is constant ($\psi' = 0$). Setting $\psi' = 0$ in the costate

[29] The control set $h \geq 0$ is unbounded above, so when $p - c(x) - \psi > 0$ the linear Hamiltonian increases without bound and admits no finite maximizer; a genuine bang-bang policy presupposes an effort ceiling $h \in [0, h_{\max}]$ (a harvesting-capacity constraint) or convex harvesting costs. The analysis here concerns the singular regime and the interior steady state.

equation and dividing by $\psi^* = p - c(x^*)$ from (6.4.34), assuming $p > c(x^*)$,

$$r\left(1 - \frac{2\,x^*}{K_0}\right) - \frac{c'(x^*)\,h^*}{p - c(x^*)} \;=\; \rho. \tag{6.4.35}$$

Equation (6.4.35) is the Clark golden rule [Clark (1990)]. It is a no-arbitrage condition, that is, a requirement that two alternative ways of holding wealth yield the same return, since otherwise the owner would abandon one for the other and the proposed path could not be optimal. Here the asset is the fish stock itself, and the two alternatives concern a marginal fish at the steady state. The first alternative is to harvest it now: it sells for the net margin $p-c(x^*) = \psi^*$, and the proceeds earn the rate ρ at which the planner discounts the future. The second alternative is to leave it in the water, which yields two sources of return. First, a biological return: the marginal fish reproduces, raising natural growth by the slope of the logistic curve, $r\,(1 - 2\,x^*/K_0)$, positive for $x^* < K_0/2$, zero at $x^* = K_0/2$, and negative beyond. Second, a cost saving: a denser stock makes every harvested fish cheaper to catch, which is worth $-\,c'(x^*)\,h^* > 0$ per unit of time, that is, $-\,c'(x^*)\,h^*/(p - c(x^*)) > 0$ per unit of value invested in the stock. At the optimum the total return per unit of value from leaving the fish in the water, the left-hand side of (6.4.35), must equal the rate ρ earned by harvesting and investing, the right-hand side: a lower return would make immediate depletion preferable, a higher one further conservation.

Phase diagram.

In the (x, h) plane, with x the state and h the control, two curves organize the dynamics. The natural-growth locus $h = r\,x\,(1-x/K_0)$

is an inverted parabola with roots 0 and K_0 and peak $h_{\mathrm{MSY}} = r\,K_0/4$ at $x = K_0/2$, the maximum sustainable yield (MSY); on it $x' = 0$, above it $x' < 0$, below it $x' > 0$. The locus $\psi' = 0$ obtained from the golden rule (6.4.35) generally involves both x and h through the cost term $c'(x)\,h$, so it is a curve in the (x, h) plane; it reduces to the vertical line $x = x^*$ only after intersecting it with $x' = 0$ to eliminate h, or, directly, in the constant-cost case $c'(x) = 0$. Because h is a control, the field has no vertical component; its horizontal component $x' = r\,x\,(1 - x/K_0) - h$ is set by the distance to the parabola. The optimum is the intersection $P^* = (x^*, h^*)$, where harvest equals growth at the golden-rule stock. With constant costs the optimal stock satisfies $x^* < K_0/2$, so the MSY stock lies to its right; with stock-dependent costs the cost-saving term can move x^* to, or beyond, the MSY stock (see Figure 6.16).

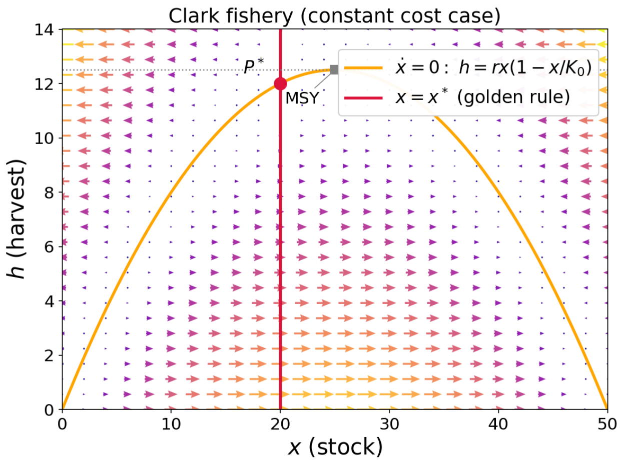

Figure 6.16. Phased diagram of the Clark model.

The integrand $F(x,h) = [\,p - c(x)\,]\,h$ is linear and unbounded in h, with $\partial F/\partial x = -\,c'(x)\,h \geq 0$ of constant sign but $\partial F/\partial h = p - c(x)$ of variable sign; F is not weakly monotone in (x,h), so Theorem 6.4.3 does not apply directly. Verifying by hand: along the candidate, $x^*(t)$ converges to x^* and $\psi^*(t)$ to $p - c(x^*)$, both finite; since the discount factor $e^{-\rho t}$ vanishes, $\lim_{t\to\infty} e^{-\rho t}\,\psi^*(t)\,x^*(t) = 0$. A full proof of global optimality, which exploits the most-rapid-approach structure of the solution, is given in Clark (1990).

Economic interpretation.

The Clark golden rule (6.4.35) extends the Hotelling rule (6.4.33) to renewable resources. If c is independent of x ($c' = 0$), it reduces to $r\,(1 - 2\,x^*/K_0) = \rho$, so $x^* = (K_0/2)\,(1 - \rho/r) < K_0/2$: impatience pushes the optimum left of MSY, onto the ascending branch with $h^* < h_{\mathrm{MSY}}$, and $\lim_{\rho\to 0^+} x^* = K_0/2$. Stock-dependent costs ($c' < 0$) add the positive cost-saving term, pulling x^* towards larger stocks, possibly past $K_0/2$; the position of x^* relative to MSY balances impatience against stock-dependent costs. When ρ is large relative to r, no interior x^* solves (6.4.35): the steady state disappears and the optimum is full depletion.

Example 6.4.10. (Nerlove-Arrow advertising model). A firm builds a stock of goodwill $G(t)$ via advertising expenditure $u(t) \geq 0$. Goodwill depreciates at rate $\delta > 0$ and accumulates linearly with advertising:

$$G'(t) \;=\; u(t) - \delta\,G(t), \qquad G(0) = G_0 \geq 0.$$

Sales depend on goodwill via a strictly concave function $S(G)$, with $S' > 0$ and $S'' < 0$. With constant net margin $m > 0$, discount rate $r > 0$, and an upper bound $u(t) \in [0, u_{\max}]$ on the rate of advertising, the firm solves

$$\begin{aligned} &\max_{u(t)\,\in\,[0,\,u_{\max}]} && \Pi = \int_0^\infty e^{-r\,t}\,[\,m\,S(G(t)) - u(t)\,]\,dt \\ &\text{s. t. :} && G'(t) = u(t) - \delta\,G(t), \\ & && G(0) = G_0. \end{aligned}$$

The state is G and the control is u. The current-value Hamiltonian is given by

$$\mathcal{H}(G, u, \psi, t) \;=\; m\,S(G) + (\psi - 1)\,u - \psi\,\delta\,G, \tag{6.4.36}$$

linear in u. The Maximum Principle gives the costate equation

$$\psi' \;=\; (r + \delta)\,\psi - m\,S'(G). \tag{6.4.37}$$

As in Example 6.4.9, the Hamiltonian is linear in the control, so the optimal advertising is bang-bang or singular, governed by the sign of the switching function $\partial\mathcal{H}/\partial u = \psi - 1$:

$$u^*(t) \;=\; \begin{cases} u_{\max} & \text{if } \psi(t) > 1, \\ 0 & \text{if } \psi(t) < 1, \\ \text{singular} & \text{if } \psi(t) = 1. \end{cases}$$

In the singular regime, where $\psi(t) = 1$ over a time interval, the costate equation (6.4.37) reduces to

$$m\,S'(G^*) \;=\; r + \delta, \tag{6.4.38}$$

which determines a unique stock G^*, since S' is strictly decreasing, provided $m\,S'(G)$ crosses the level $r + \delta$. The state equation

$G' = u - \delta\, G$, evaluated at $G = G^*$ and $G' = 0$, fixes the advertising rate $u^* = \delta\, G^*$, which we assume satisfies $0 < u^* < u_{\max}$, so that the singular control is admissible.

Phase diagram in the (G, ψ) *plane.*

Where $\psi \neq 1$, the dynamics are pinned down by the bang-bang choice of u, giving two cases:

1. If $\psi > 1$: $u = u_{\max}$, so $G' = u_{\max} - \delta\, G$. Goodwill grows whenever $G < u_{\max}/\delta$, and the costate evolves according to (6.4.37).

2. If $\psi < 1$: $u = 0$, so $G' = -\delta\, G$. Goodwill decays exponentially, and again ψ obeys (6.4.37).

The optimal trajectory consists of an initial bang-bang segment that brings the system to the singular locus $\psi = 1$ at the exact stock level $G = G^*$; once there, u switches to the maintenance level $u^* = \delta\, G^*$ and the system stays at $(G^*, 1)$ forever. Figure 6.17 shows the construction.

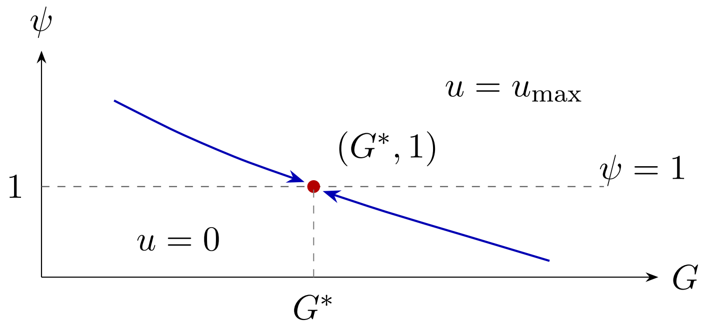

Figure 6.17. Bang-bang approach to the singular point $(G^*, 1)$ in the (G, ψ) plane.

Remark 6.4.4. The bang-bang segment must reach the singular locus $\psi = 1$ and the golden-rule stock G^* simultaneously, so that ψ crosses 1 exactly when G reaches G^*. The initial costate $\psi(0)$ is selected to enforce this matching; otherwise the trajectory either overshoots the locus $\psi = 1$ or undershoots G^*, violating optimality.

The strengthened transversality condition is verified directly. After the switching time the trajectory stays at $(G^*, 1)$, so $\psi^*(t)\, G^*(t) = G^*$ from that time onward and

$$\lim_{t\to\infty} e^{-r\,t}\,\psi^*(t)\,G^*(t) \;=\; G^* \lim_{t\to\infty} e^{-r\,t} \;=\; 0.$$

For sufficiency, the maximized Hamiltonian

$$M(G,\psi,t) \;=\; m\,S(G) - \psi\,\delta\,G + \max_{u\in[0,u_{\max}]} (\psi - 1)\,u$$

is the sum of $m\,S(G)$ (concave in G), $-\,\psi\,\delta\,G$ (linear in G), and a term independent of G, so M is concave in G. Condition (6.4.16) holds because $\psi^*(t) > 0$ along the candidate path and $G(t) \geq 0$ on any admissible path. Theorem 6.4.4 therefore applies and the candidate is the global optimum.

Economic interpretation.

Equation (6.4.38) is the Nerlove-Arrow advertising rule and admits a clean asset-pricing reading. Goodwill is a productive asset that depreciates at rate δ, and the firm's discount rate is r, so its user cost is $r + \delta$ per unit of goodwill. The marginal revenue generated by one additional unit of goodwill is $m\,S'(G)$. At the optimum the two must coincide: invest in goodwill up to the point where the marginal flow of revenue equals the marginal user cost.

The bang-bang structure of the optimal control matches empirical advertising dynamics: a firm entering a new market or launching a new product invests aggressively in brand building (the $u = u_{\max}$ phase), and once a target level of goodwill is reached, switches to a constant maintenance budget (the singular regime with $u^* = \delta\, G^*$). Both the convergence to a stationary level and the qualitative shift from "build-up" to "maintenance" are routinely documented in the marketing literature; see Nerlove and Arrow (1962) for the original formulation.

Example 6.4.11. (Tobin's Q: investment with adjustment costs). In 1981 and 1982, Summers (1981) and Hayashi (1982) studied the value-maximization problem of a competitive firm, building on James Tobin's framework [Tobin (1969)]. The firm sells all its output Y at price $p = 1$, with neoclassical technology $Y = F(K, L)$, where F is concave, $F_K > 0$, $F_L > 0$, $F_{LL} < 0$ and $F_{KK} < 0$. The firm's outlays come from three sources:

1. Labor: $w\, L$, where w is the wage. The firm takes w as given and chooses L statically: labor carries no adjustment friction and enters no intertemporal constraint, so at each instant it is set to maximize current profit, equating its marginal product to the wage, $F_L(K, L) = w$. This detaches the static choice of L from the dynamic choice of capital and leaves K as the only state variable.

2. Investment: $K' = I$ (no depreciation, for simplicity). Gross investment adds directly to the capital stock, the firm's only dynamic constraint; one unit of capital costs one unit of

output, since the investment good is the firm's own output, whose price is normalized to $p = 1$. Allowing for depreciation at rate $\delta > 0$ would replace the law of motion with $K' = I - \delta\, K$, a routine extension that does not alter the structure of the analysis.

3. Adjustment costs: $(\theta\, I^2)/(2K)$, with $\theta > 0$. These capture the additional expenses tied to implementing new investment, such as training, reorganization, and temporary disruption. The quadratic structure reflects the convex nature of these frictions: doubling the rate of investment more than doubles the disruption it generates. The division by K implies that relative adjustment costs are smaller for firms with a larger pre-existing capital stock, a standard "installed base" channel. It is precisely this convexity that makes investment a smooth, forward-looking choice rather than an indeterminate one, and gives Tobin's Q its content.

With discount rate $\rho > 0$, the firm solves

$$\begin{aligned}
\max_{I,\,L} \quad & \Pi(0) = \int_0^\infty e^{-\rho\, t} \left[F(K, L) - w\, L - I - \frac{\theta\, I^2}{2\, K} \right] dt \\
\text{s. t.}: \quad & K'(t) = I(t), \\
& K(0) = K_0 > 0.
\end{aligned}$$

The state is K and the controls are (L, I).[30] The current-value

[30] The Maximum Principle of Section 6.4 is stated for a scalar control. With several controls nothing changes except that the Hamiltonian is maximized jointly over all of them, yielding one first-order condition per control, here (6.4.39) and (6.4.40). Alternatively, since L does not enter the state equation, it can be concentrated out statically before applying the scalar theory.

Hamiltonian, with current-value costate q, is

$$\mathcal{H}(K, L, I, q, t) \;=\; F(K, L) - w\,L - I - \frac{\theta\,I^2}{2\,K} + q\,I.$$

Here the costate q is the shadow price of installed capital, the object that will become Tobin's Q. The Maximum Principle gives, for an interior solution,

$$\frac{\partial \mathcal{H}}{\partial L} \;=\; F_L(K, L) - w \;=\; 0, \tag{6.4.39}$$

$$\frac{\partial \mathcal{H}}{\partial I} \;=\; -1 - \frac{\theta\,I}{K} + q \;=\; 0, \tag{6.4.40}$$

$$q' \;=\; \rho\,q - \frac{\partial \mathcal{H}}{\partial K} \;=\; \rho\,q - F_K(K, L) - \frac{\theta}{2}\,\frac{I^2}{K^2}, \tag{6.4.41}$$

together with the transversality condition

$$\lim_{t \to \infty} e^{-\rho\,t}\,q(t)\,K(t) \;=\; 0.$$

The condition (6.4.39) fixes $L^*(K)$ statically; we focus on the dynamics of the pair (K, q).

The investment rule.

Solving (6.4.40) for I,

$$\frac{K'(t)}{K(t)} \;=\; \frac{I(t)}{K(t)} \;=\; \frac{q(t) - 1}{\theta}. \tag{6.4.42}$$

The firm invests at a rate proportional to the gap between the shadow price q and the unit replacement cost 1. If $q > 1$, an extra unit of installed capital is worth more to the firm than its replacement cost, so investment is profitable; if $q < 1$, the opposite holds and the firm disinvests. The marginal speed of adjustment is $1/\theta$: large adjustment costs (θ large) slow the response of I/K to a given gap $q - 1$.

Dividing (6.4.41) by q and rearranging,

$$\underbrace{\rho}_{\text{required return}} \;=\; \underbrace{\frac{q'}{q}}_{\text{capital gain}} \;+\; \underbrace{\frac{F_K + \frac{\theta}{2}\, I^2/K^2}{q}}_{\text{dividend yield}}.$$

The required rate of return ρ on holding capital splits into two components. The first is the capital-gain rate q'/q, the rate at which the shadow value of installed capital appreciates. The second is the dividend yield, the productive return from holding the unit, expressed in two parts: the marginal product of capital F_K (the direct flow benefit) plus $(\theta/2)\, I^2/K^2$ (the marginal saving in adjustment costs from holding a larger capital base), all normalized by q. The equation is the standard no-arbitrage condition: capital and the safe asset must deliver the same total return at every instant.

Tobin's Q as the value-to-replacement ratio.

Integrating the costate equation (6.4.41) forward and applying the transversality condition gives the marginal q (see Exercise 6.4.8):

$$q(t) \;=\; \int_t^{\infty} e^{-\rho\,(s-t)} \left[F_K + \frac{\theta}{2}\,\frac{I^2}{K^2} \right] ds. \tag{6.4.43}$$

Thus, $q(t)$ is the present discounted value of the marginal flow benefit of one extra unit of installed capital along the optimal path: the marginal product of capital plus the marginal saving in adjustment costs.

This object has an empirical counterpart. The firm's market value at date t is the present value of its profit flow,

$$\Pi(t) \;=\; \int_t^{\infty} e^{-\rho\,(s-t)} \left[F(K, L) - w\, L - I - \frac{\theta\, I^2}{2\, K} \right] ds.$$

Dividing this market value by the capital replacement cost $K(t)$ defines Tobin's average Q [Tobin (1969)]:

$$Q(t) \;=\; \frac{\Pi(t)}{K(t)} \;=\; \frac{\text{stock value of the firm}}{\text{capital replacement cost}}. \tag{6.4.44}$$

The bridge between marginal q and average Q is Hayashi's work [Hayashi (1982)]. Suppose F is homogeneous of degree one in (K, L). By Euler's theorem,[31] together with the condition $F_L = w$, this gives $F - w\,L \;=\; F_K\,K$; the adjustment cost $\theta\,I^2/(2\,K)$ is likewise homogeneous of degree one in (I, K). Along the optimal path, using also $q \;=\; 1 + \theta\,I/K$ from (6.4.40), the value $e^{-\rho\,t}\,q\,K$ evolves as

$$\frac{d}{dt}\left[\,e^{-\rho\,t}\,q(t)\,K(t)\,\right] \;=\; -\,e^{-\rho\,t}\left[F(K,L) - w\,L - I - \frac{\theta\,I^2}{2\,K}\right].$$

Integrating forward and applying the transversality condition yields $q(t)\,K(t) = \Pi(t)$, hence $q(t) = Q(t)$. Under linear homogeneity the shadow price of installed capital coincides with the ratio computed from market data, so a single observable statistic captures all the forward-looking information relevant for the firm's investment decision.

Then, replacing $I \;=\; (q - 1)\,K/\theta$ from (6.4.40) in the law of motion and in the costate equation (6.4.41) reduces the problem to

[31] Euler's theorem for homogeneous functions: if $F(x_1, \ldots, x_n)$ is differentiable and homogeneous of degree k, that is $F(\lambda\,x) = \lambda^k\,F(x)$ for every $\lambda > 0$, then $\sum_{i=1}^{n} x_i\,\partial F/\partial x_i = k\,F(x)$. In particular, under constant returns to scale $(k = 1)$, $\sum_i x_i\,\partial F/\partial x_i = F(x)$.

an autonomous planar system in (K, q):

$$K' = \frac{q-1}{\theta} K, \tag{6.4.45}$$

$$q' = \rho q - \Phi(K) - \frac{(q-1)^2}{2\theta}, \tag{6.4.46}$$

where $\Phi(K) = F_K(K, L^*(K))$ is the marginal product of capital evaluated at the optimal labor choice $L^*(K)$ of (6.4.39).

The reduced marginal product Φ inherits its monotonicity from the concavity of F. Since $L^*(K)$ solves $F_L(K, L^*(K)) = w$, differentiating this identity in K gives the labor response

$$L^{*\prime}(K) = -F_{LK}/F_{LL},$$

and the chain rule yields

$$\Phi'(K) = F_{KK} + F_{KL} L^{*\prime}(K) = F_{KK} - \frac{F_{KL}^2}{F_{LL}} = \frac{\det(\nabla^2 F)}{F_{LL}},$$

where the last identity uses $\det(\nabla^2 F) = F_{KK} F_{LL} - F_{KL}^2$. Under strict concavity of F the Hessian is negative definite, so $\det(\nabla^2 F) > 0$ and $F_{LL} < 0$, hence $\Phi'(K) < 0$: the reduced marginal product is strictly decreasing. Strict concavity is also what permits an interior steady state. Under the linear homogeneity assumed in Hayashi's theorem, by contrast, $F_L = w$ fixes the ratio L/K and Φ, homogeneous of degree zero, is constant, so no interior steady state exists at the firm level; the analysis below therefore describes the decreasing-returns case.

Since $K > 0$, Equation (6.4.45) vanishes exactly when $q = 1$: a shadow price equal to the unit replacement cost leaves no incentive to invest, so $I = 0$ and capital is stationary. The $K' = 0$ isocline is therefore the horizontal line $q = 1$.

The locus on which the shadow price is momentarily stationary is obtained by setting $q' = 0$ in (6.4.46). Rearranging,

$$\rho\, q \;=\; \Phi(K) \;+\; \frac{(q-1)^2}{2\,\theta},$$

the left-hand side is the return the firm requires on the shadow value q of one unit of capital, and the right-hand side is what that unit delivers, the marginal product $\Phi(K)$ plus the marginal saving in adjustment costs $(q-1)^2/(2\,\theta)$; the locus collects the pairs (K, q) at which the two balance.

This equation is quadratic in q but enters K only through $\Phi(K)$. Since Φ is strictly decreasing, hence invertible, it is cleanest to solve for K. Writing

$$g(q) \;=\; \rho\, q - \frac{(q-1)^2}{2\,\theta}$$

for the net required return, the condition $q' = 0$ reads $\Phi(K) = g(q)$, so that, inverting Φ,

$$K \;=\; \Phi^{-1}(\, g(q)\,)\,. \tag{6.4.47}$$

Because the marginal product is positive, $\Phi(K) > 0$ for every $K > 0$, equation (6.4.47) has a solution only where $g(q) > 0$. As a function of q, g is a downward parabola, positive on an open interval $]\, q_-, q_+ \,[$ whose endpoints are the roots of $\rho\, q = (q-1)^2/(2\,\theta)$. Since $g(1) = \rho > 0$, the value $q = 1$ lies inside this interval, so the isocline is well defined in a neighborhood of the steady state. Finally, along $K' = 0$ with $K > 0$ we must have $q = 1$, and substituting into (6.4.47) gives $\Phi(K^*) = g(1) = \rho$, hence

$$K^* \;=\; \Phi^{-1}(\rho), \qquad P^* \;=\; \big(\, \Phi^{-1}(\rho),\, 1 \,\big)\,.$$

The firm accumulates capital until its marginal product equals the discount rate.

Now, differentiating the identity $\Phi(K) = g(q)$ implicitly along the locus gives $\Phi'(K)\, dK/dq = g'(q)$, with $g'(q) = \rho - (q-1)/\theta$, so the slope in the (K, q) plane is

$$\frac{dq}{dK} = \frac{\Phi'(K)}{\rho - (q-1)/\theta}. \tag{6.4.48}$$

At P^*, where $q = 1$, the denominator is ρ, so the slope is $\Phi'(K^*)/\rho < 0$ there: the locus is strictly decreasing through the steady state. More globally, $\Phi'(K) < 0$ throughout, while $g'(q) = \rho - (q-1)/\theta$ is positive for $q < 1 + \rho\,\theta$ and negative for $q > 1+\rho\,\theta$. The locus is therefore decreasing for $q < 1+\rho\,\theta$, attains its leftmost point (a minimum of K, where g peaks and the tangent is vertical) at $q = 1 + \rho\,\theta$, and bends back for larger q. The steady state $q = 1$ sits on the decreasing branch, so in the economically relevant window with q near 1 the locus is a single-valued, strictly decreasing curve.

For the curvature, differentiate (6.4.48) once more along the locus. Writing $D(q) = \rho - (q-1)/\theta$, so that $dq/dK = \Phi'(K)/D$ and $D'(q) = -1/\theta$, the chain rule gives

$$\frac{d^2q}{dK^2} = \frac{\Phi''(K)}{D} + \frac{(\Phi'(K))^2}{\theta\, D^3}, \tag{6.4.49}$$

which at P^*, where $D = \rho > 0$, becomes

$$\left.\frac{d^2q}{dK^2}\right|_{P^*} = \frac{\Phi''(K^*)}{\rho} + \frac{(\Phi'(K^*))^2}{\theta\,\rho^3}.$$

The second term is strictly positive, so the locus is convex at P^* whenever $\Phi''(K^*) > -\left(\Phi'(K^*)\right)^2/(\theta\,\rho^2)$; in particular $\Phi'' \geq 0$

suffices. The standard technologies deliver $\Phi'' > 0$ outright. For Cobb–Douglas $F = K^\alpha L^\beta$ with $\alpha + \beta < 1$, the labor condition $F_L = w$ gives $L^*(K) = (\beta/w)^{1/(1-\beta)}\, K^{\alpha/(1-\beta)}$, and substituting into $F_K = \alpha\, K^{\alpha-1} L^\beta$ yields

$$\Phi(K) \;=\; c\, K^\gamma, \qquad \gamma = \frac{\alpha+\beta-1}{1-\beta} < 0, \qquad c = \alpha\, (\beta/w)^{\beta/(1-\beta)} > 0,$$

so that $\Phi'(K) = c\,\gamma\, K^{\gamma-1} < 0$ and $\Phi''(K) = c\,\gamma\,(\gamma-1)\, K^{\gamma-2} > 0$, since $\gamma < 0$ makes $\gamma\,(\gamma-1) > 0$.

Two distinct forces thus produce the convexity, one from technology and one from the adjustment friction. First, diminishing returns that themselves diminish, $\Phi'' > 0$: as K rises the marginal product flattens, so to keep the balance $\rho\, q = \Phi(K) + (q-1)^2/(2\,\theta)$ the shadow price q must fall ever more slowly, and the curve flattens as K grows, which is convexity; this is the first term in (6.4.49). Second, the quadratic adjustment cost contributes $(\Phi')^2/(\theta\, D^3) > 0$: even if the marginal product were locally linear ($\Phi'' = 0$), the feedback of q on itself through the $(q-1)^2$ term alone would bend the locus convex. Intuitively, the curve is steep where K is small, since there the marginal product is high and changes quickly, so a small change in K requires a large change in q to restore balance, and flat where K is large, since then the marginal product barely moves. This is exactly the shape of a decreasing convex curve.

Vector field.

The two isoclines divide the plane into regions, and the direction of motion in each is fixed by the signs of K' and q'. From (6.4.45), since $K > 0$,

$$K' > 0 \iff q > 1, \qquad K' < 0 \iff q < 1,$$

so capital grows above the line $q = 1$ and shrinks below it. For the shadow price, from (6.4.46), $\partial q'/\partial K = -\Phi'(K) > 0$, so q' increases with K at fixed q: to the right of the $q' = 0$ locus (larger K) the shadow price rises, $q' > 0$, and to its left it falls, $q' < 0$. Combining the two signs gives the arrows of Figure 6.18, which displays the direction of motion at each point together with the two isoclines. It is the vector field of the system, and shows where the state is heading from any position, but it does not by itself constitute a complete phase diagram.

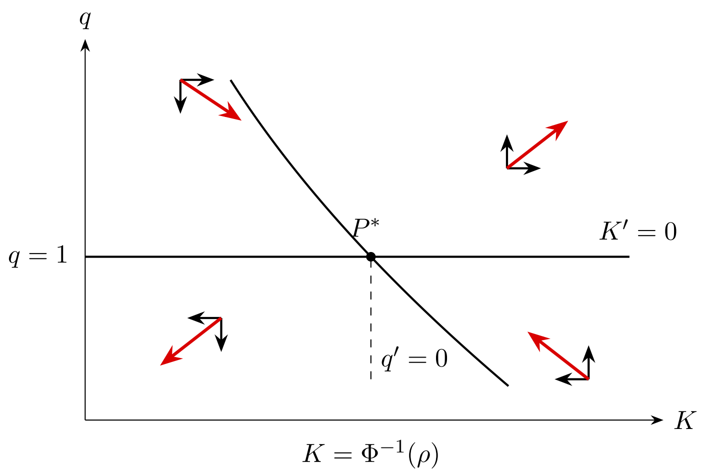

Figure 6.18. Vector field of the Tobin Q system.

Linearizing (6.4.45) and (6.4.46) at P^*,

$$J_{P^*} = \begin{bmatrix} 0 & K^*/\theta \\ -\Phi'(K^*) & \rho \end{bmatrix}, \qquad \det J_{P^*} = \frac{K^*}{\theta}\,\Phi'(K^*) < 0,$$

since $\Phi'(K^*) < 0$. A negative determinant forces real eigenvalues of opposite sign, so P^* is a saddle. Its stable eigenvector defines the saddle path, the unique trajectory converging to P^*; the saddle path fixes $q(0)$ for each given K_0, and along it the transversality condition selects the optimal solution.

Assembling these elements gives the complete phase diagram of Figure 6.19. The saddle path is everywhere downward sloping and lies close to the $q' = 0$ locus but is flatter than it, below the locus for $K < K^*$ and above it for $K > K^*$; this reflects the eigenvector slope being smaller in magnitude than the isocline slope $\Phi'(K^*)/\rho$ at P^*. The vertical dashed line marks the modified golden rule $K^* = \Phi^{-1}(\rho)$, at which the saddle path crosses $q = 1$.

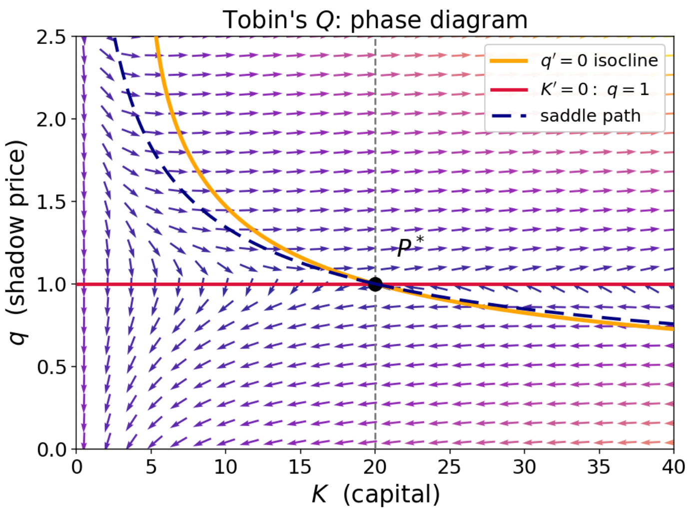

Figure 6.19. Phase diagram of the Tobin Q model.

Sufficiency and transversality.

Along the saddle path $q(t)$ converges to 1 and $K(t)$ to K^*, both finite, so the strengthened transversality condition holds by direct substitution: $\lim_{t\to\infty} e^{-\rho t}\, q(t)\, K(t) \;=\; 0.$ For sufficiency, the maximization of $\mathcal{H}$ over the two controls separates. The labor part $\max_L\, [\, F(K,L) - w\, L\,]$ is concave in K, because partial maximization of a concave function preserves concavity; the investment part $\max_I\, [\,(q-1)\, I - \theta\, I^2/(2\, K)\,] \;=\; (q-1)^2\, K/(2\,\theta)$ is linear in K. The maximized Hamiltonian is therefore concave in K. Condition (6.4.16) holds because $q(t) > 0$ along the saddle path and $K(t) > 0$ on any admissible path, so Theorem 6.4.4 applies and the saddle path is the global optimum.

Comparative statics.

Consider a permanent fall in the discount rate from ρ_{old} to $\rho_{\text{new}} < \rho_{\text{old}}$. The $K' = 0$ locus depends only on q, so it is unaffected and stays at $q = 1$. The $q' = 0$ locus passes through the steady-state capital fixed by $\Phi(K^*) = \rho$. Differentiating this identity,

$$\frac{dK^*}{d\rho} \;=\; \frac{1}{\Phi'(K^*)} \;<\; 0, \qquad \Phi'(K^*) \;=\; F_{KK} - \frac{F_{KL}^2}{F_{LL}} \;<\; 0,$$

where Φ' includes the endogenous labor response $L^{*\prime} = -F_{LK}/F_{LL}$ from $F_L(K, L^*(K)) = w$, and the sign follows from strict concavity of F. A lower discount rate therefore raises the steady-state stock, $K^*_{\text{new}} \;>\; K^*_{\text{old}}$: the $q' \;=\; 0$ locus shifts rightward while $q \;=\; 1$ is unchanged. Figure 6.20 shows the shift in the vector field.

Economic interpretation.

Tobin's Q recasts firm investment as a forward-looking decision driven by a single observable statistic: the ratio of market value to replacement cost. When $q > 1$, the market signals that an extra unit of installed capital is worth more than its replacement cost, and the firm responds by investing at rate $(q-1)/\theta$. The model is the workhorse of empirical investment regressions and underpins the modern q-theory of investment. Two properties deserve emphasis.

The investment decision depends on the entire forward path of profits only through q. This is a strong empirical implication: proxies for q should explain investment more reliably than current-period profits or marginal product estimates [Hayashi (1982)].

The comparative-static exercise on ρ in Figure 6.20 captures the long-run effect of monetary or fiscal policies that lower the cost of capital: a lower ρ raises K^* permanently, without changing the steady-state $q^* = 1$. The transition dynamics, however, generate a temporary increase of q above 1 that triggers the new investment phase, as the firm invests at rate $(q-1)/\theta > 0$ until K converges to the new K^*.

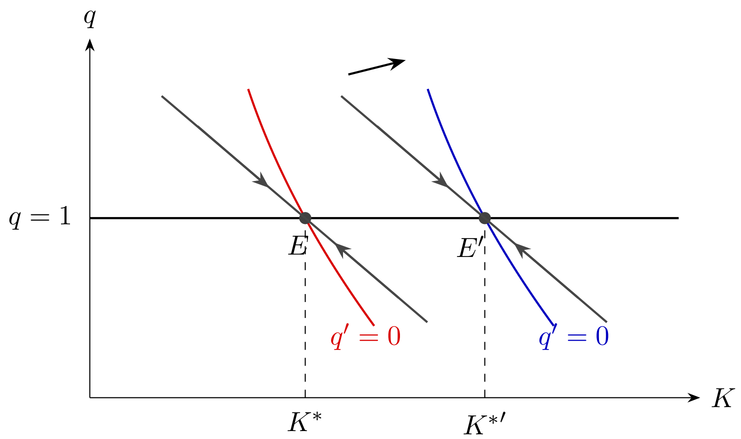

Figure 6.20. Comparative statics: a fall in ρ shifts the $q' = 0$ locus rightward (red to blue), moving the steady state from E to E'.

Example 6.4.12. (Endogenous technological change). In 1990, Paul Romer's paper "Endogenous Technological Change" Romer (1990), published in the *Journal of Political Economy*, integrated the production sector, the research and development (R&D) sector, and the household sector within a single growth framework. Romer's model microfounds growth as the optimal allocation of human capital between producing goods and producing ideas. Designs created in the R&D sector enable the production of new durable goods, which firms use as inputs in final-goods production. The economy comprises four interacting units linked in a chain. Households supply labor and human capital, own the firms, and choose how much human capital to devote to research. The final-goods sector combines labor and human capital with a range of specialized durable inputs to produce the single final good. Each durable is supplied by a monopolist who owns its design. The R&D sector produces the new designs that widen the range of available durables. Growth originates in the last link: a wider range of designs raises the productivity of physical capital, and the speed at which designs accumulate is set by the household's allocation of human capital to research.

Formally, the representative household owns the firms, supplies labor $L > 0$ inelastically, and holds the aggregate stock of human capital $H > 0$, both constant. The household chooses consumption $C(t)$ and the allocation $H_A(t) \in [0, H]$ of human capital to research, with $H_Y(t) = H - H_A(t)$ employed in goods production.

The competitive final-goods sector takes durable input prices $p(i, t)$ and wages as given and combines labor, goods-sector human capital, and a continuum of specialized durable inputs $x(i,t)$, $i \in [0, A(t)]$, into the final good through the Cobb-Douglas-with-varieties technology

$$Y(t) \;=\; H_Y(t)^{\alpha}\, L^{\beta} \int_0^{A(t)} x(i,t)^{1-\alpha-\beta}\, di, \tag{6.4.50}$$

with $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Here $A(t)$ is the stock of designs, the measure of distinct durable varieties invented by date t, and the index i runs over those varieties, so the available inputs fill the interval $[0, A(t)]$. Treating the varieties as a continuum, rather than a finite list, lets $A(t)$ grow smoothly as a state variable and turns the sum over varieties into the integral in (6.4.50). Innovation is horizontal: a new design adds a new variety and widens the range, rather than improving an existing input.

The final-goods firm chooses each durable input $x(i,t)$ to maximize instantaneous profits

$$Y(t) - w_H\, H_Y - w_L\, L - \int_0^{A(t)} p(i,t)\, x(i,t)\, di,$$

where w_H and w_L are the rental rate of human capital and the wage. The first-order condition with respect to $x(i,t)$ yields the inverse demand for each variety,

$$p(i,t) \;=\; (1-\alpha-\beta)\, H_Y^{\alpha}\, L^{\beta}\, x(i,t)^{-(\alpha+\beta)}, \tag{6.4.51}$$

a constant-elasticity demand schedule, with price elasticity $1/(\alpha + \beta)$ independent of the level of x. Each design i is patented, so its owner is the sole supplier of variety i and prices it as a monopolist in the durable-goods (capital inputs that are not used up in a single period but yield productive services over many periods) sector.The producer manufactures $x(i,t)$ from the final good (one unit of the final good yields one unit of the durable, the standard normalization), faces the inverse demand (6.4.51), and chooses the quantity to maximize instantaneous profits $\pi(x) = p(x)\,x - x$. Differentiating $\pi(x) = (1-\alpha-\beta)\,H_Y^\alpha\,L^\beta\,x^{1-(\alpha+\beta)} - x$ with respect to x and setting the derivative to zero,

$$(1-\alpha-\beta)\,p(x) \;-\; 1 \;=\; 0,$$

which gives the monopoly price as a constant markup over marginal cost,

$$p(i,t) \;=\; \frac{1}{1-\alpha-\beta}.$$

Substituting back into (6.4.51), every variety is produced in the same amount, $x(i,t) = \overline{x}(t)$ for all $i \in [0, A(t)]$, with

$$\overline{x}(t) \;=\; \left[\,(1-\alpha-\beta)^2\,H_Y(t)^\alpha\,L^\beta\,\right]^{1/(\alpha+\beta)}.$$

The monopoly flow profit, identical across designs, is $\pi(t) = (\alpha+\beta)\,\overline{x}(t)/(1-\alpha-\beta)$.

The R&D sector uses research-sector human capital H_A together with the existing stock of designs A to produce new designs at the rate

$$A'(t) \;=\; \delta\,H_A(t)\,A(t), \qquad \delta > 0, \tag{6.4.52}$$

where δ is the productivity of research. The fact that A appears multiplicatively on the right-hand side is the standing-on-shoulders externality: a larger stock of existing designs raises the productivity of further research.

Identifying the aggregate stock of physical capital with the integral of durables,

$$K(t) \;=\; \int_0^{A(t)} x(i,t)\,di \;=\; A(t)\,\overline{x}(t),$$

and substituting the symmetric quantity into (6.4.50) gives the reduced-form aggregate technology

$$Y(t) \;=\; H_Y(t)^{\alpha}\,L^{\beta}\,A(t)^{\alpha+\beta}\,K(t)^{1-\alpha-\beta}. \tag{6.4.53}$$

The technology has constant returns in (K, H_Y, L) for fixed A and increasing returns once A is included; the exponent $\alpha+\beta$ on A measures the strength of the variety externality, the channel through which a wider menu of inputs raises the productivity of physical capital.

The social planner internalizes both the standing-on-shoulders externality in (6.4.52) and the variety externality in (6.4.53). Under CRRA preferences of coefficient $\sigma > 0$, $\sigma \neq 1$ and discount rate $\rho > 0$, the planner solves

$$\begin{aligned}
\max_{C,\,H_A} \qquad & U = \int_0^{\infty} e^{-\rho t}\,\frac{C(t)^{1-\sigma}-1}{1-\sigma}\,dt \\
\text{s. t. :} \qquad & K' = (H_Y\,A)^{\alpha}\,(L\,A)^{\beta}\,K^{1-\alpha-\beta} - C, \\
& A' = \delta\,H_A\,A, \\
& H_Y + H_A = H, \\
& K(0) = K_0 > 0, \quad A(0) = A_0 > 0.
\end{aligned}$$

The economy has two state variables (K and A) and two controls (C and H_A). Let ψ and μ denote the current-value costates of K and A. The current-value Hamiltonian is

$$\mathcal{H} \;=\; \frac{C^{1-\sigma}-1}{1-\sigma} + \psi\,[\,Y - C\,] + \mu\,\delta\,H_A\,A.$$

The Maximum Principle gives, for an interior solution,

$$\frac{\partial \mathcal{H}}{\partial C} \;=\; C^{-\sigma} - \psi \;=\; 0, \tag{6.4.54}$$

$$\frac{\partial \mathcal{H}}{\partial H_A} \;=\; -\,\psi\,\alpha\,Y/H_Y + \mu\,\delta\,A \;=\; 0, \tag{6.4.55}$$

$$\psi' \;=\; \rho\,\psi - \psi\,(1-\alpha-\beta)\,Y/K, \tag{6.4.56}$$

$$\mu' \;=\; \rho\,\mu - \psi\,(\alpha+\beta)\,Y/A - \mu\,\delta\,H_A, \tag{6.4.57}$$

together with the transversality conditions

$$\lim_{t\to\infty} e^{-\rho\,t}\,\psi(t)\,K(t) \;=\; 0, \qquad \lim_{t\to\infty} e^{-\rho\,t}\,\mu(t)\,A(t) \;=\; 0.$$

Equation (6.4.55) equates the shadow value of moving one unit of human capital out of goods production (the loss is $\psi\,\alpha\,Y/H_Y$, the shadow value of forgone output) with the shadow value of moving it into research (the gain is $\mu\,\delta\,A$, the shadow value of the new designs created).

Differentiating (6.4.54) with respect to time and using (6.4.56),

$$\frac{C'(t)}{C(t)} \;=\; \frac{1}{\sigma}\left[(1-\alpha-\beta)\,\frac{Y(t)}{K(t)} - \rho\right].$$

On a BGP the research allocation is constant, $H_A(t) = H_A^*$ for all t, so the goods-sector allocation $H_Y = H - H_A^*$ is constant as well. From the design dynamics (6.4.52), the stock of designs grows at the constant rate

$$\frac{A'(t)}{A(t)} \;=\; \delta\,H_A^*,$$

which we denote by g^*. These growth rates are mutually consistent. Differentiating the reduced-form technology (6.4.53) in logarithms, with H_Y and L constant,

$$\frac{Y'}{Y} \;=\; (\alpha+\beta)\,\frac{A'}{A} + (1-\alpha-\beta)\,\frac{K'}{K}.$$

A constant consumption-to-capital ratio requires Y/K constant, that is, $Y'/Y = K'/K$; substituting this above gives $K'/K = A'/A = g^*$, and the resource constraint $K' = Y - C$ then forces $C'/C = g^*$. All four variables therefore grow at $g^* = \delta\, H_A^*$, and it remains to pin down H_A^*.

The two costates grow at a common rate. From the consumption condition (6.4.54), $\psi = C^{-\sigma}$, so

$$\frac{\psi'}{\psi} \;=\; -\,\sigma\,\frac{C'}{C} \;=\; -\,\sigma\,g^*.$$

The static condition (6.4.55) can be written $\mu = (\alpha\,\psi\,Y)/(\delta\,H_Y\,A)$; with H_Y constant and Y/A constant along the path, μ is proportional to ψ, so $\mu'/\mu = \psi'/\psi = -\sigma\,g^*$.

Dividing the costate equation (6.4.57) by μ writes it in growth form,

$$\frac{\mu'}{\mu} \;=\; \rho - (\alpha+\beta)\,\frac{Y}{A}\,\frac{\psi}{\mu} - \delta\,H_A.$$

The costate ratio is removed using (6.4.55) once more,

$$\frac{\psi}{\mu} = \frac{\delta\,H_Y\,A}{\alpha\,Y},$$

which collapses the middle term to

$$(\alpha+\beta)\,\frac{Y}{A}\,\frac{\psi}{\mu} \;=\; \frac{\alpha+\beta}{\alpha}\,\delta\,H_Y.$$

Substituting this and $\mu'/\mu = -\sigma\,g^*$ into the growth-form equation leaves a single scalar relation,

$$-\,\sigma\,g^* \;=\; \rho - \frac{\alpha+\beta}{\alpha}\,\delta\,H_Y - \delta\,H_A.$$

Replacing the two allocations by the growth rate they imply, $\delta\,H_A = g^*$ by definition and $\delta\,H_Y = \delta\,H - g^*$, and writing $\Lambda = \alpha/(\alpha+\beta)$

so that $(\alpha+\beta)/\alpha = 1/\Lambda$,

$$-\sigma\, g^* \;=\; \rho - \frac{1}{\Lambda}\,(\delta\, H - g^*) - g^*.$$

Multiplying by Λ and collecting the terms in g^* gives

$$g^*\,(\Lambda\,\sigma + 1 - \Lambda) = \delta\, H - \Lambda\,\rho,$$

that is, the closed-form growth rate

$$g^* \;=\; \frac{\delta\, H - \Lambda\,\rho}{\Lambda\,\sigma + (1-\Lambda)}, \qquad \Lambda \;=\; \frac{\alpha}{\alpha+\beta}. \tag{6.4.58}$$

For the candidate to be a genuine interior BGP, three things must hold: positive growth $g^* > 0$, an interior research allocation $H_A^* \in]0, H[$, and the two transversality conditions. From (6.4.58), $g^* > 0$ if and only if $\delta\, H > \Lambda\,\rho$, that is, $\rho < \delta\, H/\Lambda$; and $H_A^* = g^*/\delta < H$ if and only if $(1-\sigma)\,\delta\, H < \rho$. Both requirements form the single chain

$$(1-\sigma)\,\delta\, H \;<\; \rho \;<\; \frac{\delta\, H}{\Lambda}.$$

For the transversality conditions, along the BGP $\psi(t) = \psi(0)\, e^{-\sigma\, g^* t}$ and $K(t) = K_0\, e^{g^* t}$, so $e^{-\rho\, t}\,\psi(t)\, K(t)$ carries the exponent $(1-\sigma)\, g^* - \rho$; since μ is proportional to ψ and A grows at g^*, the condition for (μ, A) carries the same exponent. Within the chain $(1-\sigma)\, g^* < \rho$ holds (immediately if $\sigma \geq 1$, and from $g^* < \delta\, H$ if $\sigma < 1$), so both transversality conditions are satisfied and the discounted-utility integral converges. For sufficiency, an argument parallel to that of Example 6.4.7 applies: for each fixed H_A the state-dependent block of the Hamiltonian, $\psi\,(H_Y\, A)^\alpha\,(L\, A)^\beta\, K^{1-\alpha-\beta} + \mu\,\delta\, H_A\, A$, is concave and homogeneous of degree one in (K, A), and maximizing over $H_A \in [0, H]$ yields a maximized Hamiltonian that is concave in (K, A).

Vector field.

To visualize convergence to the balanced growth path, detrend the two states by the stock of designs and define the per-design variables $k = K/A$ and $c = C/A$. Holding the research allocation at its balanced-growth value H_A^*, so that $H_Y^* = H - H_A^*$ is constant, the reduced-form technology (6.4.53) gives $Y/A = B\,k^{1-\alpha-\beta}$ and $Y/K = B\,k^{-(\alpha+\beta)}$, where $B = (H_Y^*)^\alpha\,L^\beta$. Writing $g^* = \delta\,H_A^*$, the detrended variables obey the following dynamic

$$\begin{aligned} k' &= B\,k^{1-\alpha-\beta} - c - g^*\,k, \\ c' &= c\left[\frac{1}{\sigma}\,\big((1-\alpha-\beta)\,B\,k^{-(\alpha+\beta)} - \rho\big) - g^*\right]. \end{aligned}$$

The locus $k' = 0$ is the curve $c = B\,k^{1-\alpha-\beta} - g^*\,k$, and the locus $c' = 0$ is the vertical line $k = k^*$ fixed by

$$(1-\alpha-\beta)\,B\,(k^*)^{-(\alpha+\beta)} = \rho + \sigma\,g^*.$$

These two loci split the plane into four regions. Since k' is decreasing in c, the horizontal component of the flow points to the right below the curve $k' = 0$, where $k' > 0$, and to the left above it. Since c' is decreasing in k, the vertical component points upward to the left of $k = k^*$, where $c' > 0$, and downward to the right. In each region the resultant of these two components, shown in red in Figure 6.21, gives the direction of motion.

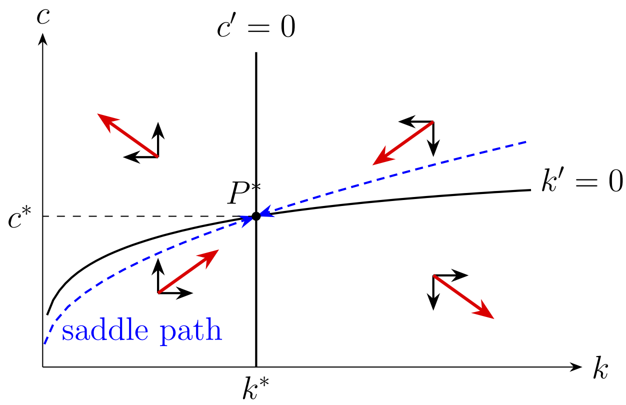

Figure 6.21. Vector field of the Romer model.

Phase diagram.

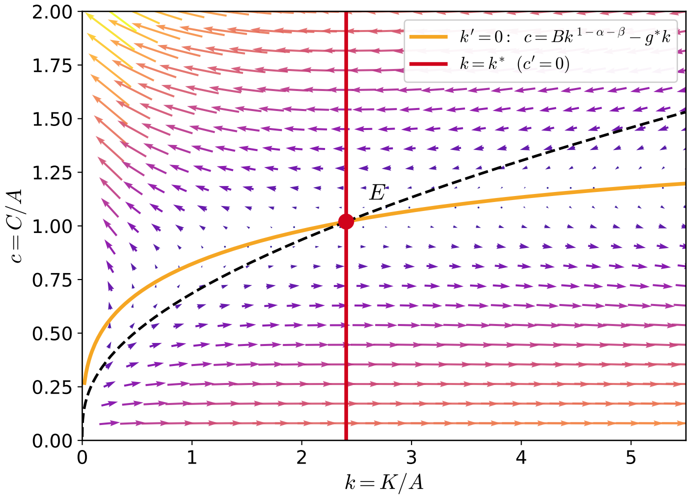

Figure 6.22. Phase diagram of the Romer model.

The two nullclines intersect at $E = (k^*, c^*)$, a saddle whose stable manifold is the saddle path, with the same structure as the RCK phase diagram. Trajectories that start on the saddle path converge to E, the detrended image of the balanced growth path, while all others diverge. Figure 6.22 shows this phase portrait, with the two loci, the saddle point E, and the saddle path along which the economy converges.

Economic interpretation.

Several conclusions stand out. The optimal allocation of human capital to research, H_A^*, is the central object of the model: it determines long-run growth through the simple proportionality $g^* = \delta H_A^*$. It arbitrates the trade-off between two productive uses of the same scarce input. A unit of H allocated to goods production contributes immediately to current output; a unit allocated to research produces designs that raise the productivity of all future output through the variety externality of (6.4.53). The balance between these two margins is set by (6.4.55), which equates the shadow value of human capital in goods production, $\psi \alpha Y/H_Y$, with the marginal value of allocating an additional unit of human capital to research, $\mu \delta A$. Long-run growth is therefore endogenous to the model: it is determined by preferences (ρ, σ) and by the technology and endowment parameters (δ, H, α, β), not by an exogenous rate of technical progress. Growth depends on the level of human capital H, not its rate of change: a larger economy in human-capital terms grows permanently faster, the celebrated scale effect of first-generation endogenous-growth models. More patient economies (lower ρ) and a higher elasticity of intertemporal substitution (lower σ) both raise growth by raising H_A^*, the human capital allocated to innovation.

Romer's model thus places the long-run determinants of prosperity in the hands of policies that affect either the productivity δ of the R&D sector or the household's incentive to allocate human capital toward research rather than toward production.

Example 6.4.13. (Government in the RCK model). We extend Example 6.4.4 by introducing a government that levies distortionary taxes,[32] makes lump-sum transfers V to households, and purchases an exogenous flow of goods and services G. The economy remains closed in goods and assets; the open-economy extension is left as an exercise.

Population grows at the constant rate $n > 0$, so that $L(t) = e^{n\,t}$ (we normalize $L(0) = 1$). Labor-augmenting technological progress occurs at the constant rate $\chi \geq 0$, so the technology factor $T(t) = e^{\chi\,t}$ solves $T' = \chi\,T$ with $T(0) = 1$. The effective labor input is $\widehat{L}(t) = L(t)\,T(t) = e^{(n+\chi)\,t}$. Throughout, as before, aggregate variables are denoted by capitals (K, L, C, A, V, G, Π), per-capita variables (divided by L) by lower case ($k = K/L$, $c = C/L$, $a = A/L$, $v = V/L$, $g = G/L$), and variables per unit of effective labor (divided by $\widehat{L}$) by a hat ($\widehat{k} = K/\widehat{L}$, $\widehat{c} = C/\widehat{L}$, $\widehat{a} = A/\widehat{L}$, $\widehat{v} = V/\widehat{L}$, $\widehat{g} = G/\widehat{L}$). The household's pure rate of

[32] A tax is distortionary when it alters the relative prices agents face, changing their marginal decisions. The taxes below are distortionary in general, but their bite depends on the active margins: with labor supplied inelastically, τ_w is effectively lump-sum, and a constant τ_c shifts the static consumption first-order condition without changing its growth rate, as the discussion after (6.4.70) shows. A lump-sum transfer V, by contrast, depends on no choice variable.

time preference is ρ, and we maintain the structural assumptions $\rho > n$ and $\rho - n > (1-\theta)\,\chi$ throughout, which together guarantee convergence of the lifetime utility integral introduced below along paths on which consumption grows at the technological rate χ. Tax rates are constants in $[0,1[$: τ_w on wages, τ_a on asset income, τ_c on consumption, and τ_f on firm profits.

The government runs a balanced budget,

$$G + V \;=\; \tau_w\, w\, L \;+\; \tau_a\, r\, A \;+\; \tau_c\, C \;+\; \tau_f\, \Pi, \tag{6.4.59}$$

where Π denotes taxable corporate profit (the operating surplus net of depreciation, made precise in the firm's problem below) and r, w are the equilibrium rental rate of capital and the wage, respectively. The policy block $\{G(t), V(t), \tau_w, \tau_a, \tau_c, \tau_f\}$ is exogenous, with one of G or V adjusting residually so that (6.4.59) holds at every $t \geq 0$. The representative household owns the aggregate stock of assets $A(t)$, measured in units of the consumption good. The flow accumulation equation states that the change in assets equals after-tax labor income, plus after-tax capital income, plus transfers, minus tax-inclusive consumption expenditure:

$$\begin{aligned} A'(t) \;=\; & (1-\tau_w)\, w(t)\, L(t) \\ & +\; (1-\tau_a)\, r(t)\, A(t) \;-\; (1+\tau_c)\, C(t) \;+\; V(t). \end{aligned} \tag{6.4.60}$$

Dividing (6.4.60) by $L(t)$ and substituting $a'(t) = A'(t)/L(t) - n\, a(t)$, we obtain

$$\begin{aligned} a'(t) \;=\; & (1-\tau_w)\, w(t) \\ & +\; \left[\, (1-\tau_a)\, r(t) - n \,\right] a(t) \;-\; (1+\tau_c)\, c(t) \;+\; v(t). \end{aligned} \tag{6.4.61}$$

The term $-n\,a$ captures the dilution of per-capita assets caused by population growth: even with A constant, a falls at rate n because the same stock is shared among more individuals.

Preferences are CRRA, $u(c) = (c^{1-\theta} - 1)/(1-\theta)$ for $\theta > 0$, $\theta \neq 1$ (and $u(c) = \ln c$ for $\theta = 1$), so that $u'(c) = c^{-\theta}$; this is the same specification as in Example 6.4.4. The household solves

$$\max_{c(t)\,\geq\,0} \quad \int_0^\infty e^{-(\rho-n)\,t}\, u(c(t))\, dt$$
$$\text{s.t.}: \quad \text{budget constraint (6.4.61), with } a(0) = a_0 \geq 0 \text{ and } a(t) \geq 0.$$

As in the classical RCK model, the effective discount rate $\rho - n$ already incorporates the adjustment from aggregate to per-capita utility: at date t each per-capita utility flow $u(c)$ is enjoyed by $L(t) = e^{n\,t}$ identical individuals, so

$$\int_0^\infty e^{-\rho\,t}\, L(t)\, u(c)\, dt \;=\; \int_0^\infty e^{-(\rho-n)\,t}\, u(c)\, dt.$$

The current-value Hamiltonian, with costate ψ, is

$$\begin{aligned}\mathcal{H}(a, c, \psi, t) \;&= u(c) + \psi\,\left[\,(1-\tau_w)\,w + (1-\tau_a)\,r\,a - (1+\tau_c)\,c\right.\\ &\qquad\qquad\qquad\qquad \left. -\,n\,a + v\,\right].\end{aligned}$$

The Pontryagin Maximum Principle delivers three conditions, which we state first and then manipulate. The static first-order condition in the control is

$$\frac{\partial \mathcal{H}}{\partial c} \;=\; c^{-\theta} \;-\; (1+\tau_c)\,\psi \;=\; 0; \tag{6.4.62}$$

the costate equation is

$$\psi'(t) \;=\; (\rho - n)\,\psi(t) \;-\; \frac{\partial \mathcal{H}}{\partial a}; \tag{6.4.63}$$

and the transversality condition, in current value, is

$$\lim_{t\to\infty} e^{-(\rho-n)\,t}\,\psi(t)\,a(t) \;=\; 0. \tag{6.4.64}$$

Substituting $\partial\mathcal{H}/\partial a = \psi\,\left[\,(1-\tau_a)\,r - n\,\right]$ into (6.4.63) gives

$$\psi'(t) \;=\; \left[\,\rho - (1-\tau_a)\,r\,\right]\,\psi(t). \tag{6.4.65}$$

Differentiating the logarithm of (6.4.62) and using (6.4.65) gives $-\theta\,c'(t)/c(t) = \psi'(t)/\psi(t) = \rho - (1-\tau_a)\,r(t)$, hence

$$\frac{c'(t)}{c(t)} \;=\; \frac{(1-\tau_a)\,r(t) - \rho}{\theta}. \tag{6.4.66}$$

Integrating (6.4.65) and inserting $\psi(t)$ into (6.4.64), the factor $e^{-\rho\,t}$ cancels and the positive constant $\psi(0)$ drops out, leaving

$$\lim_{t\to\infty} a(t)\,\exp\left(-\int_0^t \left[\,(1-\tau_a)\,r(s) - n\,\right] ds\right) \;=\; 0. \tag{6.4.67}$$

This condition rules out asset over-accumulation and, together with (6.4.66) and the budget (6.4.61), pins down the optimal path.

The representative firm operates the neoclassical, constant-returns technology $Y = F(K,\widehat{L})$, with F in C^2, strictly increasing in each argument and homogeneous of degree one; writing $\widehat{k} = K/\widehat{L}$ and $f(\widehat{k}) = F(\widehat{k},1)$, we assume f is strictly concave with $f' > 0$. Capital is rented at r and labor at w. The rental cost $r\,K$ is not deductible from the corporate tax base, so taxable corporate profit is the operating surplus net of depreciation,

$$\Pi \;=\; F(K,\widehat{L}) \;-\; w\,L \;-\; \delta\,K;$$

the firm is taxed at rate τ_f on Π and maximizes its net-of-tax payoff $(1-\tau_f)\,\Pi - r\,K$, which is zero in equilibrium. Constant returns give

$F_K = f'(\widehat{k})$ and $F_{\widehat{L}} = f(\widehat{k}) - \widehat{k}\, f'(\widehat{k})$, so the first-order conditions in K and L read

$$f'(\widehat{k}) \;=\; \frac{r}{1-\tau_f} \;+\; \delta, \qquad w \;=\; e^{\chi t}\left[\, f(\widehat{k}) - \widehat{k}\, f'(\widehat{k}) \right]. \quad (6.4.68)$$

Remark 6.4.5. A higher τ_f raises the effective cost of capital, since $r\,K$ is not deductible. The firm breaks even only at a higher $f'(\widehat{k})$, hence a lower $\widehat{k}$ and, by (6.4.68), a lower wage, hurting workers even though they are not directly taxed on profits.

Asset-market clearing $A = K$ gives $\widehat{a} = \widehat{k}$. The resource constraint $Y = C + I + G$ with $I = K' + \delta\, K$ and $\widehat{L}'/\widehat{L} = n + \chi$ yields

$$\widehat{k}'(t) \;=\; f(\widehat{k}) \;-\; \widehat{c} \;-\; (\chi + n + \delta)\,\widehat{k} \;-\; \widehat{g}. \quad (6.4.69)$$

Substituting $r = (1-\tau_f)\left[\, f'(\widehat{k}) - \delta \right]$ from (6.4.68) into (6.4.66) and using $\widehat{c} = c\, e^{-\chi t}$ (so $\widehat{c}'/\widehat{c} = c'/c - \chi$),

$$\frac{\widehat{c}'(t)}{\widehat{c}(t)} \;=\; \frac{1}{\theta}\left[(1-\tau_a)\,(1-\tau_f)\,\Big(f'(\widehat{k}) - \delta \Big) \;-\; \rho \;-\; \theta\,\chi \right]. \quad (6.4.70)$$

The balanced-growth path is the steady state $(\widehat{k}^*, \widehat{c}^*)$ of the system (6.4.69) and (6.4.70). Setting $\widehat{c}' = 0$ gives

$$f'(\widehat{k}^*) \;=\; \delta \;+\; \frac{\rho + \theta\,\chi}{(1-\tau_a)\,(1-\tau_f)},$$

and $\widehat{k}' = 0$ gives

$$\widehat{c}^* \;=\; f(\widehat{k}^*) - (\chi + n + \delta)\,\widehat{k}^* - \widehat{g}.$$

On the path $\widehat{k}, \widehat{c}$ are constant, while k, c grow at χ and the aggregates K, C, Y at $n + \chi$.

Vector field.

The nullclines $\widehat{k}' = 0$ and $\widehat{c}' = 0$, the curves on which each detrended variable is stationary, are the curve $\widehat{c} = f(\widehat{k}) - (\chi + n + \delta)\widehat{k} - \widehat{g}$ and the vertical line $\widehat{k} = \widehat{k}^*$. They split the plane into four regions. Since $\widehat{k}'$ is decreasing in $\widehat{c}$, the horizontal component of the flow points to the right below the $\widehat{k}' = 0$ curve and to the left above it. Since the after-tax marginal product falls in $\widehat{k}$, $\widehat{c}'$ is decreasing in $\widehat{k}$, so the vertical component points upward to the left of $\widehat{k}^*$ and downward to the right. In each region the resultant of the two components, shown in red in Figure 6.23, gives the direction of motion.

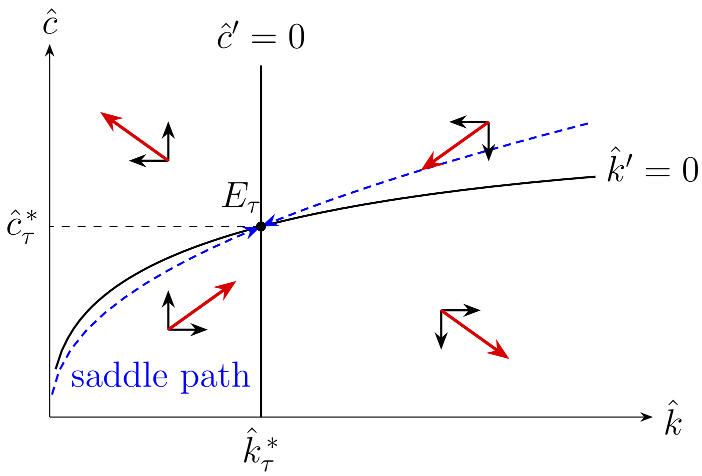

Figure 6.23. Vector field of the taxed RCK model.

Phase diagram.

The two nullclines intersect at $E_\tau = (\widehat{k}_\tau^*, \widehat{c}_\tau^*)$, a saddle whose stable manifold is the saddle path; trajectories that start on it converge to E_τ, while all others diverge. The portrait has the same saddle-path structure as the base RCK model. Relative to the undistorted benchmark E_0 (all taxes set to zero), distortionary taxes shift the $\widehat{c}' = 0$ line to the left and move the saddle point to E_τ, with lower $\widehat{k}^*$ and $\widehat{c}^*$ (Figure 6.24).

Along this path r is constant with $(1 - \tau_a)\, r = \rho + \theta\, \chi$, and a grows at rate χ, so the exponent in (6.4.67) is $\chi + n - (1 - \tau_a)\, r = -\,[\,(\rho - n) - (1 - \theta)\, \chi\,] < 0$ by the structural assumptions, and the transversality condition holds.

Both τ_a and τ_f lower the after-tax return in (6.4.70), reducing $\widehat{k}^*$. The wage tax τ_w and the consumption tax τ_c do not enter (6.4.70): with inelastic labor supply τ_w is effectively lump-sum, and a constant τ_c shifts the level of marginal utility but not its growth rate. Government spending G enters only the resource constraint, so it is pure waste in this benchmark.

The structure of (6.4.69) and (6.4.70) mirrors Example 6.4.4, and the link is no accident. Setting $\tau_w = \tau_a = \tau_c = \tau_f = 0$ and $G = V = \chi = 0$, the system collapses exactly to that of Example 6.4.4: with $A = K$ and the firm's conditions (6.4.68), the decentralized economy reproduces the planner's allocation, as guaranteed under the standard conditions of the welfare theorems (complete markets, price taking, convexity, no externalities; see Mas-Colell et al. (1995), Chapters 16 and 17). Distortionary taxes break this equivalence: the private after-tax return governing the Euler equation (6.4.66) is the social net return $f'(\widehat{k}) - \delta$ times the wedge $(1 - \tau_a)\,(1 - \tau_f) < 1$.

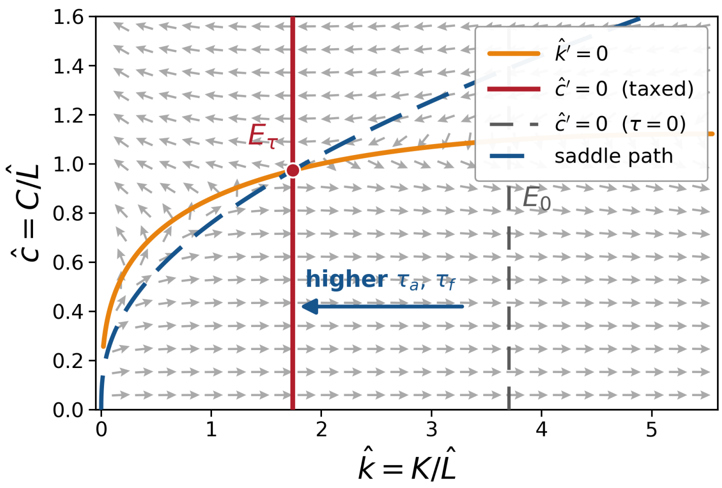

Figure 6.24. Phase diagram of the taxed RCK model.

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This concludes the theoretical development and the worked examples of the chapter. We hope the reader now appreciates how deeply optimal control theory is woven into modern dynamic economic analysis, and into macroeconomic dynamics and growth theory in particular. Every model above, from the canonical RCK framework to the endogenous-growth and resource-extraction examples, is ultimately a special instance of the single problem $\mathcal{P}^{\infty}_{c,\rho}$, solved by one recurring procedure: form the current-value Hamiltonian, apply the Maximum Principle to obtain the optimality and costate conditions, impose the transversality condition, and confirm optimality through a concavity (sufficiency) argument. What changes from one application to the next is only the economic content poured into the state, the control, and the technology; the underlying machinery, developed in Sections 6.1–6.4, remains unchanged. The appendix turns from macroeconomics to a problem of mechanism design, showing that the same tools are equally at home in microeconomic theory, not only in growth and aggregate dynamics.

PROBLEM SET

Exercise 6.4.1. (Convergence of the discounted objective).

(a) Let $F(x,u,t) = e^{-\rho t}\,G(x,u)$ with $\rho > 0$, and let (x,u) be an admissible pair along which the running benefit grows at most exponentially: there exist constants $M > 0$ and $\gamma \in \mathbb{R}$ such that

$$|\,G\,(x(t),u(t))\,| \;\leq\; M\,e^{\gamma\,t}, \qquad \forall\,t \geq t_0.$$

Show that $J = \int_{t_0}^{\infty} F(x(t),u(t),t)\,dt$ is absolutely convergent whenever $\rho > \gamma$, and that

$$|\,J\,| \;\leq\; \frac{M\,e^{-(\rho-\gamma)\,t_0}}{\rho-\gamma}.$$

(b) Specialize to a CRRA utility $u(c) = \left(c^{1-\theta}-1\right)/(1-\theta)$, $\theta > 0$, along a path $c(t) \sim c_0\,e^{g\,t}$. Show that $\int_0^{\infty} e^{-\rho\,t}\,u(c(t))\,dt$ converges if and only if $\rho > \max\{\,0,(1-\theta)\,g\,\}$.

<u>Hint</u>: Apply part (a) with $\gamma = \max\{\,0,(1-\theta)\,g\,\}$ to obtain sufficiency, then evaluate the two integrals explicitly to obtain the necessary condition.

(c) With $\psi = c^{-\theta}$ and a state $x \sim e^{g\,t}$, show that the transversality condition $\lim_{t\to\infty} e^{-\rho\,t}\,\psi(t)\,x(t) = 0$ holds if and only if $\rho > (1-\theta)\,g$.

(d) Apply (b) and (c) to the Lucas (Example 6.4.7) and Romer (Example 6.4.12) models, and verify the sufficient condition. Does it holds?

Exercise 6.4.2. (Verifying the strengthened transversality condition). Consider a model in which the optimal trajectories on the BGP take the form

$$x^*(t) \;=\; x_0\,e^{g\,t}, \qquad \psi^*(t) \;=\; \psi_0\,e^{\eta\,t},$$

for some constants $g, \eta \in \mathbb{R}$, $x_0, \psi_0 > 0$, and a discount rate $\rho > 0$.

a) Show that the strengthened transversality condition

$$\lim_{t\to\infty} e^{-\rho\,t}\,\psi^*(t)\,x^*(t) \;=\; 0$$

holds if and only if

$$g + \eta \;<\; \rho.$$

b) In the Hotelling problem of Example 6.4.8 with $c = 0$ and $S^*(t) \to 0$, the candidate is $S^*(t) = S_0\,e^{-r\,t}$ and $\psi^*(t) = \psi_0\,e^{r\,t}$. Verify the inequality of part (a) directly.

Exercise 6.4.3. (AK model with capital-income taxation). Reconsider Example 6.4.6. Suppose now that the government levies a constant tax $\tau \in [0, 1[$ on the gross capital return A, and rebates the proceeds back to the household as a lump-sum transfer $V(t)$. The household's budget per capita is

$$k'(t) \;=\; [\,(1-\tau)\,A - \delta - n\,]\;k(t) \;-\; c(t) \;+\; v(t),$$

with $v = V/L$ and government balanced budget $V = \tau\,A\,K$ at every t. Preferences and the rest of the environment are as in Example 6.4.6.

a) Write down the current-value Hamiltonian, the FOC for consumption, and the costate equation.

b) Show that the Euler equation (6.4.25) becomes
$$\frac{c'(t)}{c(t)} \;=\; \frac{(1-\tau)\,A - \delta - \rho}{\theta} \;=\; g(\tau).$$

c) State the Lucas condition for the modified problem and discuss how the restriction varies with τ.

d) Compute $dg/d\tau$. By how much does long-run growth fall in percentage points when τ rises from 0 to 0.25, assuming $A = 0.10$, $\delta = 0.05$, $\rho = 0.02$, $\theta = 1$ (log utility)?

Exercise 6.4.4. (Lucas model: log utility and no depreciation). Specialize the Lucas human-capital model (Example 6.4.7) by setting $\theta = 1$ (log utility) and $\delta = 0$. Maintain $\xi > \rho > 0$.

a) Show that the BGP growth rate (6.4.31) reduces to $g^* = \xi - \rho$. Interpret: under log utility the income and substitution effects of the return on human capital offset each other; explain why the growth rate then involves only the schooling productivity ξ and the discount rate ρ.

b) Show that the Lucas condition (6.4.32) collapses to $\rho > 0$ in this special case. Why is this the weakest possible restriction on the discount rate?

c) Write down the closed-form time paths $c^*(t)$, $k^*(t)$, and $h^*(t)$ along the BGP, in terms of the (yet to be determined) initial values c_0, k_0, h_0, and u^*. Verify that all three grow at the rate g^*.

d) Compute the optimal time allocation u^*.

   Hint: On the BGP, h grows at rate g^*, and the human-capital law of motion $h'(t) = \xi\,(1 - u(t))\,h(t)$ implies $\xi\,(1 - u^*) = g^*$. Combine with part (a).

e) Discuss qualitatively how u^* responds to a rise in ξ (more productive schools) and to a rise in ρ (more impatient households). Are the signs of these comparative statics consistent with intuition?

Exercise 6.4.5. (Hotelling rule with stock-dependent extraction cost). Modify the extraction problem of Example 6.4.8 by allowing the unit extraction cost to depend on the remaining stock,

$$c \;=\; c(S), \qquad c \in C^2, \quad c'(S) \;<\; 0, \quad c''(S) \;\geq\; 0.$$

Demand is linear: $p(q) = a - b\,q$. The discount rate is $r > 0$.

a) Set up the problem and write down the current-value Hamiltonian.

b) Show that the costate equation now reads

$$\psi'(t) \;=\; r\,\psi(t) \;+\; c'(S(t))\,q(t).$$

Hint: $\partial\mathcal{H}/\partial S = -\,c'(S)\,q$.

c) Derive the modified Hotelling rule

$$\frac{\psi'(t)}{\psi(t)} \;=\; r \;+\; \frac{c'(S(t))\,q(t)}{\psi(t)},$$

and interpret the second term on the right-hand side as a "stock-effect correction" to the standard rule (6.4.33).

d) Show that, when $c'(S) < 0$, the in-situ shadow value grows at a rate less than r. Explain economically: leaving a marginal unit in the ground keeps the stock larger and future extraction cheaper; why does this cost-saving dividend reduce the rate at which ψ must appreciate?

e) Take the limiting case $c'(S) \to 0$ and verify that the classical Hotelling rule (6.4.33) is recovered.

Exercise 6.4.6. (Clark fishery with Gompertz growth). Replace the logistic growth law in Example 6.4.9 by the Gompertz law,

$$x'(t) \;=\; r\,x(t)\,\ln\left(\frac{K_0}{x(t)}\right) \;-\; h(t), \qquad r > 0,\; K_0 > 0,$$

keeping the rest of the model unchanged: stock-independent extraction cost c, constant price p, discount rate $\rho > 0$.

a) Plot the natural-growth function $\mathcal{G}(x) = r\,x\,\ln(K_0/x)$ on $], K_0]$ and identify the maximum sustainable yield.

<u>Hint</u>: Maximize $\mathcal{G}$ over x.

b) Derive the costate equation and the singular-arc condition. Show that the Clark golden rule for this growth law is

$$r\left[\ln\left(\frac{K_0}{x^*}\right) - 1\right] = \rho.$$

<u>Hint</u>: Differentiate $\mathcal{G}$ at the steady state.

c) Solve the previous equation explicitly for x^* and show that the long-run stock is

$$x^* = K_0 \exp\left(-1 - \rho/r\right).$$

d) Compare x^* with the maximum-sustainable-yield level $x_{\mathrm{MSY}} = K_0/e$ (the latter is the unique maximizer of $\mathcal{G}$). Show that the optimum lies strictly below MSY for $\rho > 0$, and converges to MSY as $\rho \to 0$.

Exercise 6.4.7. (Nerlove-Arrow with convex advertising cost). Modify Example 6.4.10 by replacing the linear advertising expenditure $-u$ with a convex one:

$$\text{flow profits} = m\,S(G) - \frac{u^2}{2}.$$

The control is only required to be non-negative, $u \in [0, \infty[$; no upper bound is imposed. All other features of the model are unchanged.

a) Write down the current-value Hamiltonian. Why is the optimal control no longer of bang-bang type?

b) Derive the FOC and show that $u^*(t) = \psi(t)$. Interpret: the firm advertises at a rate equal to the shadow value of an extra unit of goodwill.

c) Eliminate u from the state equation to obtain the planar system $G' = \psi - \delta\, G$, $\psi' = (r + \delta)\, \psi - m\, S'(G)$. Sketch the isoclines $G' = 0$ and $\psi' = 0$ in the (G, ψ) plane.

d) Show that the unique steady state (G^*, ψ^*) satisfies $m\, S'(G^*) = (r + \delta)\, \psi^*$ and $\psi^* = \delta\, G^*$. Combine these to obtain a single equation for G^*.

e) Linearize the system at (G^*, ψ^*) and verify that the steady state is a saddle, with a one-dimensional stable manifold in the (G, ψ) plane.

Exercise 6.4.8. (Forward representation of marginal q). In the setting of Example 6.4.11, integrate the costate equation (6.4.41) forward and use the transversality condition to show that the marginal shadow price admits the forward integral representation

$$q(t) \;=\; \int_t^{\infty} e^{-\rho\,(s-t)} \left[F_K + \frac{\theta}{2}\, \frac{I^2}{K^2} \right] ds,$$

with the integrand evaluated along the optimal path.

Hint: multiply (6.4.41) by the integrating factor $e^{-\rho\, t}$ and note that $(d/dt)\,(e^{-\rho\, t}\, q) = e^{-\rho\, t}\; (q' - \rho\, q)$.

Appendix 6.A Applications in contract theory and additional proofs

6.A.1 The Principal-Agent Model

The previous sections of this chapter developed optimal control theory in the context of dynamic economic problems, where time was the running parameter. In this subsection we show that the same theory provides the natural language for an entirely static class of microeconomic problems: those of contract theory under hidden information, also known as mechanism design with adverse selection. References for this topic include Mirrlees (1971), Tadelis and Segal (2005), Laffont and Martimort (2002), Salanié (2005) and Börgers (2015). The connection that this appendix presents is that, when the agent's private information takes values in a continuum, the principal's optimization is a one-dimensional optimal control problem in which the agent's type, θ, plays the role formerly played by time. Pontryagin's Maximum Principle (Theorem 6.1.1) yields the celebrated Mirrlees-Mussa-Rosen screening solution.

We consider a risk-neutral principal (a monopolistic seller, a regulator, or a tax authority, depending on the application) that contracts with a single agent whose private characteristic is $\theta \in \Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$. We refer to θ as the agent's type. The principal does not observe θ, but holds prior beliefs described by an absolutely continuous distribution with strictly positive density $f(\theta) > 0$ and cumulative distribution function $F(\theta)$ on Θ.

A contract is a pair of functions $(x, T) : \Theta \to X \times \mathbb{R}$, where $x(\theta) \in X \subset \mathbb{R}_+$ is the allocation (a quantity of a good, a quality level, an effort, a production volume, etc.) and $T(\theta) \in \mathbb{R}$ is the transfer the agent pays to the principal.[33] The agent's preferences are quasilinear,

$$v(x, \theta) - T,$$

with $v \in C^2(X \times \Theta)$ satisfying

$$v_x(x, \theta) > 0, \qquad v_{xx}(x, \theta) < 0, \qquad v(0, \theta) = 0, \tag{6.A.1}$$

and the principal incurs a C^1 cost $c(x)$ with $c'(x) > 0$ and $c''(x) \geq 0$. The agent's outside option yields utility 0.

The structural assumption that organizes the entire theory is single crossing: a higher-type agent values a marginal increase in the allocation strictly more than a lower-type agent. Formally, we have the following definition.

Definition 6.A.1. The agent's value function $v(x, \theta)$ satisfies the single-crossing property (SCP), also known as the Spence-Mirrlees condition, if

$$v_{x\theta}(x, \theta) \ > \ 0 \qquad \forall\ (x, \theta) \in X \times \Theta. \tag{6.A.2}$$

The terminology comes from the geometry of indifference curves in the (x, T)-plane: under SCP, the indifference curves of two

[33] Throughout this appendix we reserve the symbol t for time, in keeping with the notation of the rest of the chapter, and use the upper-case $T(\theta)$ for the transfer schedule. The reader who consults the contract-theory literature will find in some works $t(\theta)$ used instead.

distinct types cross at most once. The economic content is that the marginal willingness to pay for the allocation, $v_x(x, \theta)$, is strictly increasing in θ.

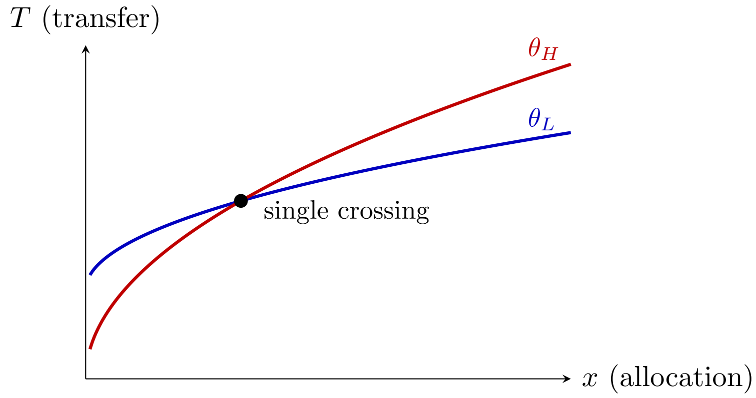

Figure 6.25. Indifference curves of two types $\theta_L < \theta_H$ in the (x, T)-plane.

The reason SCP is so powerful is that it forces any choice problem parametrized by θ to have a monotone solution. The following classical result, due to Topkis (1998) and sharpened by Edlin and Shannon (1998), is the workhorse of monotone comparative statics. We state it in the form needed for this appendix; see Milgrom and Shannon (1994) for a general treatment.

Theorem 6.A.1. (Topkis; Edlin-Shannon). Let $\varphi : X \times \Theta \to \mathbb{R}$ be a C^2 function, where $X \subset \mathbb{R}$ and $\Theta \subset \mathbb{R}$ are intervals, and assume $\varphi_{x\theta}(x, \theta) > 0$ for all (x, θ). For every $\theta \in \Theta$, let $x^*(\theta) \in \arg\max_{x \in X} \varphi(x, \theta)$ be a maximizer, assumed to exist. Then $x^*(\theta)$ is non-decreasing in θ. Moreover, if the maximizer is interior to X for two distinct values $\theta' > \theta$, then $x^*(\theta') > x^*(\theta)$.

Proof. Let $\theta' > \theta$ and let $x' = x^*(\theta')$, $x = x^*(\theta)$ be the corresponding maximizers. Optimality gives

$$\varphi(x,\theta) \ \geq\ \varphi(x',\theta), \qquad \varphi(x',\theta') \ \geq\ \varphi(x,\theta'),$$

and adding these two inequalities gives

$$\varphi(x',\theta') - \varphi(x,\theta') \ \geq\ \varphi(x',\theta) - \varphi(x,\theta).$$

Since $\varphi_{x\theta} > 0$, the function φ has strictly increasing differences: if $x' < x$, then $\varphi(x,\cdot) - \varphi(x',\cdot)$ is strictly increasing in θ, which reverses the displayed inequality strictly, a contradiction. Hence $x' \geq x$. For the strict version, suppose $x' = x$ is interior for both parameter values; the interior first-order condition gives $\varphi_x(x,\theta) = \varphi_x(x,\theta') = 0$, which is impossible because $\varphi_x(x,\cdot)$ is strictly increasing in θ by $\varphi_{x\theta} > 0$. □

We will apply Theorem 6.A.1 repeatedly: first to the full-information benchmark, then to the monotonicity of the optimal allocation in any incentive-compatible contract (Lemma 6.A.1), and finally to the monotonicity of the relaxed allocation under the hazard-rate condition discussed at the end of the appendix.

THE FULL INFORMATION BENCHMARK

Suppose, hypothetically, that the principal observed θ. It would then offer a bundle $(x(\theta), T(\theta))$ solving

$$\mathcal{P}_{\mathrm{FB}}: \quad \begin{cases} \displaystyle\max_{(x,T)\in X\times\mathbb{R}} & T - c(x) \\ \text{s.t.}: & v(x,\theta) - T \ \geq\ 0 \quad (\text{IR}). \end{cases}$$

The individual-rationality (or participation) constraint (IR) binds at the optimum; otherwise, the principal could increase T without violating it. Substituting $T = v(x,\theta)$ into the objective reduces $\mathcal{P}_{\mathrm{FB}}$[34] to the unconstrained problem

$$x^{\mathrm{FB}}(\theta) \in \arg\max_{x\in X} \left[\, v(x,\theta) - c(x) \,\right],$$

i.e., the principal extracts all surplus and implements the total-surplus-maximizing allocation, which by single-crossing and Theorem 6.A.1 is non-decreasing in θ. This is the benchmark against which the second-best (asymmetric-information) solution will be compared.

HIDDEN INFORMATION: FINITELY MANY TYPES

Suppose the type space is finite, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ with $\theta_1 < \theta_2 < \cdots \theta_n$, and let $\pi_i = \Pr(\theta = \theta_i) \in \,]0,1[$, $\sum_{i=1}^n \pi_i = 1$. A direct contract is a menu $\{(x_i, T_i)\}_{i=1}^n$, with type θ_i choosing the bundle (x_i, T_i). The contract must satisfy

$$(\mathrm{IR}_i) \quad v(x_i,\theta_i) - T_i \;\geq\; 0, \qquad i = 1,\ldots,n, \tag{6.A.3}$$

$$(\mathrm{IC}_{ij}) \quad v(x_i,\theta_i) - T_i \;\geq\; v(x_j,\theta_i) - T_j, \qquad \forall\, i \neq j. \tag{6.A.4}$$

There are n individual-rationality constraints (6.A.3) and $n(n-1)$ incentive-compatibility constraints (6.A.4); the simplest non-trivial case is $n = 2$, with $(\theta_1, \theta_2) = (\theta_L, \theta_H)$.

The constraints (6.A.3) ensure voluntary participation type by type, and (6.A.4) ensure truthful revelation: type-θ_i should weakly prefer its assigned bundle to that of any other type θ_j.

[34] FB stands for first best.

Without (IC), there is no reason for the agent to reveal θ truthfully; with (IC), the principal's design screens the n types into distinct (or sometimes coincident) bundles. The number of constraints in (6.A.4) grows quadratically in n, but under single-crossing the optimum has a much sparser structure: Maskin and Riley (1984) show that, in the regular case (when the relaxed solution is monotone, so no bunching occurs), only $n-1$ of the (6.A.4) bind at the optimum, namely the downward-adjacent ones

$$v(x_i,\theta_i) - T_i \;\geq\; v(x_{i-1},\theta_i) - T_{i-1}, \qquad i = 2,\dots,n, \tag{6.A.5}$$

and the participation constraint binds only at the lowest type, IR_1. Intuitively, by single-crossing each type's only credible deviation is to mimic the type immediately below it, and once (6.A.5) holds, all upward and longer-range deviations are ruled out automatically. The optimum exhibits five qualitative features that carry over to the continuum limit treated next:

1. Monotonicity: $x_1 \leq x_2 \leq \cdots \leq x_n$ — higher types receive (weakly) larger allocations, the incentive-compatibility requirement that pins down implementability.

2. No rent at the bottom: $v(x_1,\theta_1) - T_1 = 0$ — the lowest type is held to its participation constraint, which binds and is the only one that does.

3. Information rents above the bottom: $v(x_i,\theta_i) - T_i > 0$ for every $i \geq 2$ — every type above the lowest earns a strictly positive surplus, the rent it extracts from its private information.

4. No distortion at the top: $x_n = x^{\mathrm{FB}}(\theta_n)$ — the highest type is served efficiently, since distorting it would forgo surplus without slackening any binding constraint.

5. Downward distortion below the top: $x_i < x^{\mathrm{FB}}(\theta_i)$ for $i = 1, \dots, n-1$ — all lower types are allocated below the first-best, the principal trading efficiency for a reduction in the rents paid to the types above them.

CONTINUUM OF TYPES

We now turn to the case $\Theta = [\underline{\theta}, \overline{\theta}]$. A direct contract is a pair of functions $(x(\theta), T(\theta))$ with $x : \Theta \to X$ piecewise C^1 and $T : \Theta \to \mathbb{R}$ piecewise C^1. The continuum analogues of (6.A.3) and (6.A.4) are

$$(\mathrm{IR}_\theta) \quad v(x(\theta), \theta) - T(\theta) \ \geq \ 0, \qquad \forall\, \theta \in \Theta, \tag{6.A.6}$$

$$(\mathrm{IC}_{\theta,\widehat{\theta}}) \quad v(x(\theta), \theta) - T(\theta) \ \geq \ v(x(\widehat{\theta}), \theta) - T(\widehat{\theta}), \qquad \forall\, \theta, \widehat{\theta} \in \Theta. \tag{6.A.7}$$

Although the principal could in principle commit to far more general contracts, the Revelation Principle [(Mas-Colell et al., 1995, Section 23.C)] guarantees that restricting attention to direct schedules $(x(\theta), T(\theta))$ satisfying (6.A.6)–(6.A.7) entails no loss of generality. The set of (IC) constraints is now a double continuum, which appears intractable. The technique, due to Mirrlees (1971), is to replace (6.A.7) by an equivalent pair of conditions involving a single-variable differential equation and a monotonicity requirement.

A first observation, whose proof runs parallel to that of Theorem 6.A.1, is that the optimal allocation in any incentive-compatible contract is non-decreasing.

Lemma 6.A.1. Let $(x(\theta), T(\theta))$ satisfy (6.A.7) on Θ. Then $x(\theta)$ is non-decreasing in θ.

Proof. Take any $\theta', \theta \in \Theta$ with $\theta' > \theta$. Writing the (IC) constraints in both directions gives

$$\begin{aligned} v(x(\theta),\theta) - T(\theta) \;&\geq\; v(x(\theta'),\theta) - T(\theta'), \\ v(x(\theta'),\theta') - T(\theta') \;&\geq\; v(x(\theta),\theta') - T(\theta). \end{aligned}$$

Adding the two inequalities and canceling the transfers gives

$$[\, v(x(\theta'),\theta') - v(x(\theta),\theta') \,] \;\geq\; [\, v(x(\theta'),\theta) - v(x(\theta),\theta) \,] \,,$$

which can be rewritten, by the fundamental theorem of calculus, as

$$\int_{\theta}^{\theta'} [\, v_\theta(x(\theta'),s) - v_\theta(x(\theta),s) \,]\, ds \;\geq\; 0. \tag{6.A.8}$$

Suppose for contradiction that $x(\theta') < x(\theta)$. By single-crossing (6.A.2), v_θ is strictly increasing in x, so $v_\theta(x(\theta'),s) < v_\theta(x(\theta),s)$ for every $s \in [\theta, \theta']$, and the integrand in (6.A.8) is strictly negative. This contradicts (6.A.8). Hence $x(\theta') \geq x(\theta)$. $\square$

We henceforth write the monotonicity requirement in differential form, where almost every (a.e.) is meant in the sense of Lebesgue measure:[35]

$$x'(\theta) \;\geq\; 0 \quad \text{for almost every } \theta \in \Theta. \tag{M}$$

[35] A property holds almost everywhere if it fails only on a set of Lebesgue measure zero.

Now fix an incentive-compatible contract $(x(\theta), T(\theta))$ and introduce the agent's equilibrium utility

$$U(\theta) \;=\; v(x(\theta),\theta) - T(\theta). \tag{6.A.9}$$

Type θ's problem of choosing the optimal report $\widehat{\theta}$ amounts to

$$\widehat{\theta} \;\in\; \arg\max_{\widehat{\theta}\in\Theta} \Psi(\widehat{\theta},\theta), \qquad \Psi(\widehat{\theta},\theta) \;=\; v(x(\widehat{\theta}),\theta) - T(\widehat{\theta}). \tag{6.A.10}$$

At every θ at which x and T are differentiable, truth-telling implies the local first-order condition

$$\frac{\partial \Psi}{\partial \widehat{\theta}}(\theta,\theta) \;=\; v_x(x(\theta),\theta)\, x'(\theta) - T'(\theta) \;=\; 0. \tag{ICFOC}$$

Differentiating $U(\theta) = \Psi(\theta,\theta)$ in θ and applying the envelope theorem (the partial derivative with respect to the report $\widehat{\theta}$, evaluated at $\widehat{\theta} = \theta$, vanishes by (ICFOC)) gives

$$U'(\theta) \;=\; \frac{\partial \Psi}{\partial \theta}(\theta,\theta) \;=\; v_\theta(x(\theta),\theta) \quad \text{for a.e. } \theta \in \Theta. \tag{6.A.11}$$

Equation (6.A.11) is the Mirrlees envelope condition. Geometrically, it says that the rate at which the agent's information rent grows with type equals the marginal rate at which the value function rewards higher types for a given allocation.

So far we have shown that any IC contract satisfies (M) and (ICFOC). The next theorem establishes the remarkable fact that the converse also holds: monotonicity together with the local FOC suffices for global incentive compatibility.

Theorem 6.A.2. (Mirrlees characterization of IC). A continuous, piecewise C^1 contract $(x(\theta), T(\theta))$ satisfies (6.A.7) if and only if both of the following hold:

1. $x'(\theta) \geq 0$ for a.e. $\theta \in \Theta$; (M)

2. $v_x(x(\theta), \theta)\, x'(\theta) - T'(\theta) = 0$ for a.e. $\theta \in \Theta$. (ICFOC)

Proof. The only if part is established by Lemma 6.A.1 and equation (ICFOC). We prove the if part. Assume (M) and (ICFOC) hold; we must show that the function $\Psi(\widehat{\theta}, \theta)$ defined in (6.A.10) attains its maximum in $\widehat{\theta}$ at $\widehat{\theta} = \theta$, for every $\theta \in \Theta$.

Differentiating Ψ with respect to $\widehat{\theta}$ at any point of differentiability gives

$$\frac{\partial \Psi}{\partial \widehat{\theta}}(\widehat{\theta}, \theta) \;=\; v_x(x(\widehat{\theta}), \theta)\, x'(\widehat{\theta}) \;-\; T'(\widehat{\theta}).$$

Substituting $T'(\widehat{\theta}) = v_x(x(\widehat{\theta}), \widehat{\theta})\, x'(\widehat{\theta})$ from (ICFOC) applied at $\widehat{\theta}$ gives

$$\frac{\partial \Psi}{\partial \widehat{\theta}}(\widehat{\theta}, \theta) \;=\; \Big[v_x(x(\widehat{\theta}), \theta) \;-\; v_x(x(\widehat{\theta}), \widehat{\theta}) \Big]\, x'(\widehat{\theta}). \tag{6.A.12}$$

By (M), $x'(\widehat{\theta}) \geq 0$. By single-crossing (6.A.2), $v_x(x, \cdot)$ is strictly increasing in its second argument; hence the bracket in (6.A.12) is non-positive when $\widehat{\theta} > \theta$ and non-negative when $\widehat{\theta} < \theta$. Consequently

$$\frac{\partial \Psi}{\partial \widehat{\theta}}(\widehat{\theta}, \theta) \;\leq\; 0 \;\text{ for } \widehat{\theta} > \theta, \qquad \frac{\partial \Psi}{\partial \widehat{\theta}}(\widehat{\theta}, \theta) \;\geq\; 0 \;\text{ for } \widehat{\theta} < \theta.$$

Therefore $\Psi(\cdot, \theta)$ is non-decreasing on $[\underline{\theta}, \theta]$ and non-increasing on $[\theta, \overline{\theta}]$, and attains its global maximum at $\widehat{\theta} = \theta$. Since this holds for every θ, (6.A.7) is satisfied. □

Lemma 6.A.2. At any solution $(x(\theta), T(\theta))$ of the principal's profit-maximization problem subject to (6.A.6)–(6.A.7), the rent

U of (6.A.9) satisfies

$$U(\underline{\theta}) \;=\; 0,$$

and U is non-decreasing, so $U(\theta) \geq 0$ for every $\theta \in \Theta$; moreover $U'(\theta) = v_\theta(x(\theta), \theta) > 0$ at almost every θ with $x(\theta) > 0$.

Proof. We argue in two steps: first that U is non-decreasing on Θ (so that the only IR constraint that can bind is the one at $\underline{\theta}$); then that this constraint binds with equality.

By the envelope condition (6.A.11), see Milgrom and Segal (2002),

$$U'(\theta) \;=\; v_\theta(x(\theta), \theta) \quad \text{a.e. } \theta.$$

The assumption $v(0, \theta) = 0$ for all θ implies, by differentiating the constant function $\theta \mapsto v(0, \theta)$, that $v_\theta(0, \theta) = 0$. Combined with single-crossing $v_{x\theta} > 0$, the fundamental theorem of calculus gives

$$v_\theta(x, \theta) \;=\; v_\theta(0, \theta) + \int_0^x v_{x\theta}(s, \theta)\, ds \;=\; \int_0^x v_{x\theta}(s, \theta)\, ds \;\geq\; 0, \tag{6.A.13}$$

with strict inequality whenever $x > 0$. Therefore $U'(\theta) \geq 0$, with $U'(\theta) > 0$ at θ with $x(\theta) > 0$, and U is non-decreasing on Θ.

Since U is non-decreasing, the family of constraints (6.A.6) reduces to the single condition $U(\underline{\theta}) \geq 0$. If $U(\underline{\theta}) > 0$ at the optimum, the principal could increase every transfer $T(\theta)$ uniformly by a small constant $\epsilon \in\,]0, U(\underline{\theta})[$. This leaves the IC inequalities (6.A.7) unchanged (only differences of transfers across types matter for IC), preserves $U(\theta) - \epsilon \geq 0$, and raises profits by ϵ in expectation, a contradiction. Hence $U(\underline{\theta}) = 0$. □

The lemma identifies the lowest type's participation constraint as the one that binds and pins down the rent level (other types may also earn zero rent, for instance where $x(\theta) = 0$), and it shows why the agent's rent grows monotonically with type. With $U(\underline{\theta}) = 0$ in hand, integrating the envelope condition (6.A.11) from $\underline{\theta}$ up to θ yields the closed-form expression for the information rent:

$$U(\theta) \;=\; U(\underline{\theta}) \;+\; \int_{\underline{\theta}}^{\theta} U'(s)\,ds \;=\; \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s)\,ds. \tag{6.A.14}$$

Equation (6.A.14) is consistent with the defining identity $U(\theta) = v(x(\theta), \theta) - T(\theta)$ from (6.A.9): the latter is the definition of the equilibrium rent, the former its characterization as a single-crossing premium accumulated from the bottom of the type distribution. Equating the two and solving for the transfer gives

$$T(\theta) \;=\; v(x(\theta), \theta) \;-\; \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s)\,ds. \tag{6.A.15}$$

<u>THE RELAXED PROBLEM AND VIRTUAL SURPLUS</u>

The principal's expected profit is

$$\Pi \;=\; \mathbb{E}_\theta[\,T(\theta) - c(x(\theta))\,] \;=\; \int_{\underline{\theta}}^{\overline{\theta}} [\,T(\theta) - c(x(\theta))\,]\, f(\theta)\,d\theta.$$

Substituting $T(\theta) = v(x(\theta), \theta) - U(\theta)$ from (6.A.9) and using Theorem 6.A.2 to replace the (IC) constraints by the pair (M)-(ICFOC), the principal's program becomes

$$\begin{cases} \max\limits_{x} & \displaystyle\int_{\underline{\theta}}^{\overline{\theta}} \left[v(x(\theta), \theta) - c(x(\theta)) - \left(\int_{\underline{\theta}}^{\theta} v_\theta(x(s), s)\,ds \right) \right] f(\theta)\,d\theta \\ \text{s.t. :} & x(\theta) \in X, \quad x'(\theta) \geq 0, \end{cases} \tag{6.A.16}$$

where we used the closed-form expression (6.A.14) for U. Integration by parts gives

$$
\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\, f(\theta)\, d\theta \;&=\; \big[\, U(\theta)\, F(\theta)\,\big]_{\underline{\theta}}^{\overline{\theta}} \;-\; \int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)\, F(\theta)\, d\theta \\
&=\; U(\overline{\theta}) \cdot 1 \;-\; 0 \;-\; \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta), \theta)\, F(\theta)\, d\theta \\
&=\; \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta), \theta)\, d\theta \;-\; \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta), \theta)\, F(\theta)\, d\theta \\
&=\; \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta), \theta)\; (1 - F(\theta))\, d\theta, \qquad (6.A.17)
\end{aligned}
$$

where the second line used $F(\overline{\theta}) = 1$, $F(\underline{\theta}) = 0$ together with $U' = v_\theta$, and the following line replaced $U(\overline{\theta})$ by its expression $\int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(s), s)\, ds$ (renaming the dummy variable $s \to \theta$).

Substituting (6.A.17) into the principal's objective (6.A.16), the program, relaxed by dropping the monotonicity constraint (M) (to be verified ex post), reduces to

$$
\max_{x} \;\; \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left[v(x(\theta), \theta) - c(x(\theta)) \;-\; v_\theta(x(\theta), \theta)\, \frac{1 - F(\theta)}{f(\theta)} \right]}_{\text{virtual surplus}} f(\theta)\, d\theta. \tag{6.A.18}
$$

The integrand depends on x only pointwise in θ, so the problem decomposes into a continuum of pointwise maximizations: for almost every $\theta \in \Theta$,

$$
x^*(\theta) \;\in\; \arg\max_{x \in X} \left[v(x, \theta) - c(x) \;-\; \frac{1 - F(\theta)}{f(\theta)}\, v_\theta(x, \theta) \right]. \tag{6.A.19}
$$

The interior first-order condition gives

$$
v_x(x^*(\theta), \theta) \;-\; c'(x^*(\theta)) \;=\; \frac{1 - F(\theta)}{f(\theta)}\, v_{x\theta}(x^*(\theta), \theta). \tag{6.A.20}
$$

This is the virtual-surplus condition of Mirrlees (1971) and Mussa and Rosen (1978), the cornerstone of static screening theory.

Multiplying both sides of (6.A.20) by $f(\theta)$ gives

$$f(\theta)\,\big[\,v_x(x^*(\theta),\theta) - c'(x^*(\theta))\,\big] \;=\; (1 - F(\theta))\,v_{x\theta}(x^*(\theta),\theta). \tag{6.A.21}$$

The two sides have a clear interpretation. The left hand side (LHS) is the marginal total surplus generated when type θ receives an infinitesimal increase in $x(\theta)$, weighted by the density of that type. The right hand side (RHS) is the additional rent that this increase forces the principal to concede to all higher types $\theta' > \theta$ (whose total mass is $1 - F(\theta)$), weighted by the single-crossing premium $v_{x\theta}$. At the optimum the principal extracts surplus from each type θ only up to the point at which an additional unit of $x(\theta)$ would leak an equal amount of rent upward. The classical conclusions follow immediately: no distortion at the top ($v_x = c'$ at $\theta = \overline{\theta}$, since $1 - F(\overline{\theta}) = 0$); downward distortion below the top ($v_x > c'$ for every $\theta < \overline{\theta}$, hence $x^*(\theta) < x^{\mathrm{FB}}(\theta)$, with the wedge proportional to the inverse hazard rate).

<u>OPTIMAL CONTROL FORMULATION</u>

The relaxed program (6.A.16) has the structure of a one-dimensional optimal control problem in which the agent's type $\theta \in [\underline{\theta}, \overline{\theta}]$ plays the role formerly played by time. Take the agent's equilibrium utility $U(\theta)$ as the state variable and the allocation $x(\theta)$ as the control. The Mirrlees envelope condition (6.A.11) plays the role of the state equation, and Lemma 6.A.2 fixes the initial condition: $U'(\theta) = v_\theta(x(\theta),\theta)$ and $U(\underline{\theta}) = 0$, with $U(\overline{\theta})$ left free.

Writing $T = v - U$ in the principal's profit, the running benefit is $[\, v(x,\theta) - c(x) - U \,]\, f(\theta)$, and the screening program takes the canonical optimal control form

$$\mathcal{P}_S : \quad \begin{cases} \max\limits_{x(\theta)\in X,\quad x'(\theta)\geq 0} & \displaystyle\int_{\underline{\theta}}^{\overline{\theta}} [\, v(x(\theta),\theta) - c(x(\theta)) - U(\theta) \,]\, f(\theta)\, d\theta \\ \text{s.t. :} & U'(\theta) = v_\theta(x(\theta),\theta), \\ & U(\underline{\theta}) = 0 \\ & U(\overline{\theta}) \text{ free} \end{cases} \tag{6.A.22}$$

with state $U(\theta)$ and control $x(\theta)$. As before, we drop the monotonicity constraint $x'(\theta) \geq 0$ and verify it ex post. The correspondence with the dynamic problem $\mathcal{P}_c$ of the previous sections is summarized below.

Concept	**Dynamic problem** $\mathcal{P}_c$	**Screening problem**
Running parameter	time $t \in [t_0, t_1]$	type $\theta \in [\underline{\theta}, \overline{\theta}]$
State variable	$x(t)$	information rent $U(\theta)$
Control variable	$u(t) \in \Omega$	allocation $x(\theta) \in X$
State equation	$x'(t) = f(x,u,t)$	$U'(\theta) = v_\theta(x(\theta),\theta)$
Initial condition	$x(t_0) = x_0$	$U(\underline{\theta}) = 0$
Running benefit	$F(x,u,t)$	$[\, v(x,\theta) - c(x) - U \,]\, f(\theta)$
Terminal state	free or constrained	$U(\overline{\theta})$ free

The Hamiltonian, with costate $\lambda(\theta)$ associated with the state equation $U' = v_\theta(x, \theta)$, is

$$H(U, x, \lambda, \theta) \;=\; [\, v(x, \theta) - c(x) - U \,]\, f(\theta) \;+\; \lambda\, v_\theta(x, \theta). \quad (6.A.23)$$

Because the state U enters the running benefit directly, with coefficient $-f(\theta)$, the derivative $\partial H/\partial U = -f(\theta)$ is nonzero, so the costate is not constant along the optimal path; it obeys

$$\lambda'(\theta) \;=\; -\frac{\partial H}{\partial U} \;=\; f(\theta). \quad (6.A.24)$$

Since $U(\overline{\theta})$ is left free, Theorem 6.1 of Section 6.3 supplies the transversality condition $\lambda(\overline{\theta}) = 0$, and integrating (6.A.24) backward from $\overline{\theta}$ gives

$$\lambda(\theta) \;=\; -\int_{\theta}^{\overline{\theta}} f(s)\, ds \;=\; -\left(1 - F(\theta)\right). \quad (6.A.25)$$

Thus λ rises from $\lambda(\underline{\theta}) = -1$ to $\lambda(\overline{\theta}) = 0$. It varies with θ precisely because each marginal unit of rent granted at type θ must, by incentive compatibility, be granted to every higher type as well, and the mass of those higher types, $1 - F(\theta)$, shrinks at rate $f(\theta)$ as θ increases. The interior maximization condition $\partial H/\partial x = 0$ reads as follows

$$\frac{\partial H}{\partial x} \;=\; [\, v_x(x, \theta) - c'(x) \,]\, f(\theta) \;+\; \lambda(\theta)\, v_{x\theta}(x, \theta) \;=\; 0. \quad (6.A.26)$$

Substituting $\lambda(\theta) = -\left(1 - F(\theta)\right)$ from (6.A.25) into (6.A.26) gives

$$[\, v_x(x^*(\theta), \theta) - c'(x^*(\theta)) \,]\, f(\theta) \;-\; (1 - F(\theta))\, v_{x\theta}(x^*(\theta), \theta) \;=\; 0, \quad (6.A.27)$$

and dividing through by $f(\theta) > 0$ yields

$$v_x(x^*(\theta), \theta) \;-\; c'(x^*(\theta)) \;=\; \frac{1 - F(\theta)}{f(\theta)}\, v_{x\theta}(x^*(\theta), \theta),$$

which is exactly the virtual-surplus condition (6.A.20). This makes precise the sense in which the optimal-control formulation is the integration-by-parts derivation in disguise. The costate $\lambda(\theta) = -(1 - F(\theta))$ is the shadow value, in units of expected profit, of an infinitesimal relaxation of the rent at type θ: it equals (minus) the mass of types above θ, since by incentive compatibility any rent conceded at θ must be passed on to all higher types. Normalizing by the local density gives the per-type shadow cost $-\lambda(\theta)/f(\theta) = (1-F(\theta))/f(\theta) = 1/h(\theta)$, the inverse hazard rate that emerged from the integration by parts in (6.A.17), where $h(\theta) = f(\theta)/(1-F(\theta))$.

The construction above presumes that the relaxed allocation $x^*(\theta)$ of (6.A.20) is itself non-decreasing, so that the monotonicity constraint (M) holds automatically and may be ignored. A sufficient condition is the monotone hazard rate property, $[(1 - F)/f]'(\theta) \leq 0$, together with $v_{x\theta\theta} \leq 0$; the hazard-rate condition holds for the uniform, normal, exponential, and many other standard distributions. When this property fails, the pointwise maximizer $x^*(\theta)$ of (6.A.19) need not be non-decreasing. If it does decrease on some subinterval of Θ, it violates (M), so it is the allocation of no incentive-compatible contract, since by Lemma 6.A.1 every implementable allocation is non-decreasing. The second-best schedule is then recovered by ironing, the procedure of replacing the relaxed solution, on each interval where it would otherwise decrease, by a constant value chosen so that the resulting schedule is non-decreasing and the constrained objective (6.A.18) remains maximized. Geometrically, the construction replaces the integrated virtual surplus by its convex envelope; differentiating the envelope yields a non-decreasing schedule that coincides with x^* wherever x^* is increasing and is constant on each interval where x^* would otherwise decrease, so that on each such interval every type receives the same allocation, and hence the same bundle, and the types pool.

This pooling of an interval of types at a common bundle is known as bunching, and the name ironing reflects that a schedule which would otherwise be non-monotone is flattened into a monotone one. We refer the reader to Mussa and Rosen (1978) and (Tadelis and Segal, 2005, §1.6.2) for the ironing construction.

The model treated here is the simplest non-trivial case of mechanism design with adverse selection: a one-dimensional type, a single agent, a risk-neutral principal, quasilinear preferences, a static interaction, an exogenously fixed outside option, and observable allocations. Each of these assumptions has been relaxed in a substantial literature. Multidimensional types replace single-crossing by a system of partial differential equations, and bunching becomes generic (Rochet and Choné (1998)). With several independent agents the screening problem becomes the optimal-auction problem, governed by the same virtual-valuation construct that appears in (6.A.18) (Myerson (1981)). When the agent's type evolves over time and the principal contracts repeatedly, the envelope condition becomes a dynamic-envelope formula, and the analogy with optimal control of this appendix becomes literal (Pavan et al. (2014)). Book-length treatments are Laffont and Martimort (2002) and Börgers (2015).

6.A.2 Proof of Lemma 6.2.1

Proof. We apply the envelope theorem of Milgrom and Segal (2002) (their Theorem 1) along arbitrary directions, as suggested by their footnote 7 for parameters in normed vector spaces. The correspondence between their setup and ours is summarized in Table 6.2.

Throughout the proof, $\boldsymbol{\lambda}$ and t are fixed and suppressed from the notation. Fix a direction $\mathbf{d} \in \mathbb{R}^n$ and consider the auxiliary problem with scalar parameter $s \in]-\delta, \delta[$ for some $\delta > 0$:

$$\begin{aligned}\tilde{f}(\mathbf{u}, s) &= H(\mathbf{x} + s\,\mathbf{d}, \mathbf{u}),\\ \tilde{V}(s) &= \sup_{\mathbf{u}\in\Omega} \tilde{f}(\mathbf{u}, s) \;=\; H^0(\mathbf{x} + s\,\mathbf{d}).\end{aligned}$$

The hypotheses of Theorem 1 of Milgrom and Segal (2002) translate to $\tilde{V}$ at $s = 0$ as follows:

(a) $\mathbf{u}^*$ maximizes $\tilde{f}(\cdot, 0)$, since by (6.2.9)

$$\tilde{f}(\mathbf{u}^*, 0) = H(\mathbf{x}, \mathbf{u}^*) = H^0(\mathbf{x}) = \tilde{V}(0).$$

(b) By the chain rule, $\tilde{f}_s(\mathbf{u}^*, 0) = \dfrac{\partial H}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{u}^*) \cdot \mathbf{d}$, which exists by hypothesis.

(c) $\tilde{V}$ is differentiable at $s = 0$ with $\tilde{V}'(0) = \dfrac{\partial H^0}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{d}$, by the differentiability of H^0 at $\mathbf{x}$.

Concept	**Milgrom–Segal (2002)**	**Present lemma**
Choice set	X (arbitrary)	$\Omega \subset \mathbb{R}^p$
Choice variable	$x \in X$	$\mathbf{u} \in \Omega$
Parameter	$t \in [0, 1]$	$\mathbf{x} \in \mathbb{R}^n$
Objective	$f(x, t)$	$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t)$
Value function	$V(t) = \sup_{x \in X} f(x, t)$	$H^0(\mathbf{x}, \boldsymbol{\lambda}, t)$
Maximizer	$x^* \in X^*(t)$	$\mathbf{u}^* \in \Omega^*(\mathbf{x}, \boldsymbol{\lambda}, t)$
Hypothesis 1	$f_t(x^*, t)$ exists	$\partial H/\partial \mathbf{x}$ exists at $(\mathbf{x}, \mathbf{u}^*)$
Hypothesis 2	V differentiable at t	H^0 differentiable at $\mathbf{x}$
Conclusion	$V'(t) = f_t(x^*, t)$	$\partial H^0/\partial \mathbf{x} = \partial H/\partial \mathbf{x}\,(\mathbf{x}, \mathbf{u}^*)$

Table 6.2. Correspondence between Theorem 1 of Milgrom and Segal (2002) and Lemma 6.2.1.

By Theorem 1 of Milgrom and Segal (2002), $\tilde{V}'(0) = \tilde{f}_s(\mathbf{u}^*, 0)$, hence

$$\frac{\partial H^0}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{d} \;=\; \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{u}^*) \cdot \mathbf{d}.$$

Since this identity holds for every $\mathbf{d} \in \mathbb{R}^n$, the two gradient vectors coincide, which establishes (6.2.10). □

6.A.3 Deferred proofs and technical results for Section 6.4

This appendix collects the technical material omitted from Section 6.4: the precise asymptotic notions behind the terminal and transversality conditions, the value function together with the Principle of Optimality and the Hamilton–Jacobi–Bellman equation, the proofs of the necessary and sufficient conditions, and the continuity of the optimal control.

LIMIT INFERIOR AND LIMIT SUPERIOR

In the main text the terminal condition and the transversality conditions are written with ordinary limits, which is correct for the trajectories met in the applications. For a general admissible trajectory the relevant quantities need not converge, and the rigorous statements use the limit inferior and the limit superior. We recall their definition and explain why they are the right objects. For $g : [t_0, \infty) \to \mathbb{R}$, set $m(T) = \inf \{ g(t) : t \geq T \}$ and $M(T) = \sup \{ g(t) : t \geq T \}$. As T grows both suprema and infima are taken over a smaller set, so m is non-decreasing and M is non-increasing, and the limits

$$\liminf_{t\to\infty} g(t) \;=\; \lim_{T\to\infty} m(T) \;=\; \sup_{T\geq t_0} \inf_{t\geq T} g(t),$$

$$\limsup_{t\to\infty} g(t) \;=\; \lim_{T\to\infty} M(T) \;=\; \inf_{T\geq t_0} \sup_{t\geq T} g(t)$$

always exist in $[-\infty, \infty]$. One always has $\liminf_{t\to\infty} g(t) \leq \limsup_{t\to\infty} g(t)$, and the ordinary limit exists if and only if the two coincide.

Along a general admissible trajectory the product $\beta(t)\, x(t)$ may oscillate and have no limit. Requiring the limit to exist would needlessly shrink the admissible class. The condition

$$\liminf_{t\to\infty} \beta(t)\, x(t) \;\geq\; x_1$$

captures exactly an asymptotic floor: the trajectory must eventually remain above every level strictly below x_1. Replacing lim inf by lim sup would instead yield a condition too weak, satisfied by trajectories that fall arbitrarily far below x_1 infinitely often. This is the rigorous form of the terminal condition in (6.4.1); when the limit exists, as in all applications, lim inf and lim agree.

The feasibility condition (6.4.16) of the sufficiency theorem and the boundary term at infinity in its proof are also stated rigorously with lim inf and lim sup; the main-text versions with lim are the specializations valid when the relevant limits exist.

VALUE FUNCTION AND PRINCIPLE OF OPTIMALITY

Recall the value function defined in (6.4.6),

$$J(\xi, t) \;=\; \sup_{u \in \mathcal{U}_{t,\xi}} \int_t^\infty F(x(s), u(s), s)\, ds.$$

Its first property is the Principle of Optimality: an optimal plan, restarted at any later date from the state it has reached, remains optimal for the remaining problem. Beyond its intrinsic interest, this is the engine behind the Hamilton–Jacobi–Bellman equation.

Lemma 6.A.3. (Principle of Optimality). Suppose (x^*, u^*) solves the $\mathcal{P}_c^\infty$ problem. Then, for every $\tau \geq t_0$,

$$J(x_0, t_0) \;=\; \int_{t_0}^{\tau} F(x^*(s), u^*(s), s)\, ds + J(x^*(\tau), \tau). \tag{6.A.28}$$

Proof. By additivity of the integral,

$$J(x_0, t_0) \;=\; \int_{t_0}^{\tau} F(x^*, u^*, s)\, ds + \int_{\tau}^{\infty} F(x^*, u^*, s)\, ds.$$

It therefore suffices to show that

$$J(x^*(\tau),\tau) \;=\; \int_{\tau}^{\infty} F(x^*(s),u^*(s),s)\,ds. \tag{6.A.29}$$

By definition of J, the left-hand side is the supremum of $\int_{\tau}^{\infty} F(x,u,s)\,ds$ over admissible pairs (x,u) starting from $x^*(\tau)$ at time τ, so

$$J(x^*(\tau),\tau) \;\geq\; \int_{\tau}^{\infty} F(x^*,u^*,s)\,ds. \tag{6.A.30}$$

Suppose, towards a contradiction, that (6.A.30) is strict. Then there exists an admissible pair $(\tilde{x},\tilde{u})$ on $[\tau,\infty[$ with $\tilde{x}(\tau)=x^*(\tau)$ and

$$\int_{\tau}^{\infty} F(\tilde{x},\tilde{u},s)\,ds \;>\; \int_{\tau}^{\infty} F(x^*,u^*,s)\,ds.$$

Define the concatenation

$$\hat{u}(t) = \begin{cases} u^*(t), & t\in[t_0,\tau],\\ \tilde{u}(t), & t>\tau, \end{cases} \qquad \hat{x}(t) = \begin{cases} x^*(t), & t\in[t_0,\tau],\\ \tilde{x}(t), & t>\tau. \end{cases}$$

Since $\tilde{x}(\tau)=x^*(\tau)$, the pair $(\hat{x},\hat{u})$ is admissible for the $\mathcal{P}_c^\infty$ problem, and

$$\int_{t_0}^{\infty} F(\hat{x},\hat{u},s)\,ds \;>\; \int_{t_0}^{\infty} F(x^*,u^*,s)\,ds = J(x_0,t_0),$$

contradicting the optimality of (x^*,u^*). This proves (6.A.29), and hence the lemma. □

THE HAMILTON–JACOBI–BELLMAN EQUATION

Differentiating the identity of Lemma 6.A.3 along the optimal path converts the Principle of Optimality into a partial differential equation for the value function. Evaluated along the optimum, it ties the time derivative J_t to the Hamiltonian, and this link is exactly what delivers the transversality condition.

Theorem 6.A.3. (Hamilton–Jacobi–Bellman equation). Suppose the hypotheses of Theorem 6.4.1 hold, and let $\mathcal{N}$ be a neighborhood of the set $\{(x^*(t), t) : t \geq t_0\}$ on which the value function $J(\xi, t)$ in (6.4.6) is of class C^1. Then, for every $(\xi, t) \in \mathcal{N}$ such that the problem with initial condition $x(t) = \xi$ admits an optimal pair, the value function satisfies the Hamilton–Jacobi–Bellman equation

$$0 = \max_{u(t)\in\Omega} \left\{ F(\xi, u(t), t) + \frac{\partial J}{\partial t}(\xi, t) + \frac{\partial J}{\partial \xi}(\xi, t)\, f(\xi, u(t), t) \right\},$$

and the maximum is attained at $u(t) = u^*(t)$ whenever $\xi = x^*(t)$. In particular, along the optimal pair (x^*, u^*) and for almost every $t \geq t_0$,

$$F(x^*(t), u^*(t), t) + \frac{\partial J}{\partial t}(x^*(t), t) + \frac{\partial J}{\partial \xi}(x^*(t), t)\, f(x^*(t), u^*(t), t) = 0. \tag{6.A.31}$$

Proof. By Lemma 6.A.3,

$$J(x_0, t_0) \;=\; \int_{t_0}^{t} F(x^*(s), u^*(s), s)\, ds + J(x^*(t), t), \qquad t \geq t_0.$$

The left-hand side is constant in t. Differentiating both sides with respect to t, and using the chain rule together with $(x^*)' = f(x^*, u^*, t)$, gives

$$0 \;=\; F(x^*, u^*, t) + J_t(x^*, t) + J_\xi(x^*, t)\, f(x^*, u^*, t),$$

which is (6.A.31). Now consider any admissible control u for the initial condition $x(t) = \xi$ that is constant on $[t, t+h]$, and let x

denote the state path it generates with $x(t) = \xi$. By the definition of the value function,

$$J(\xi, t) \;\geq\; \int_t^{t+h} F(x(s), u(t), s)\, ds + J(x(t+h), t+h);$$

subtracting $J(\xi, t)$, dividing by $h > 0$ and letting $h \to 0^+$ gives

$$F(\xi, u(t), t) + \frac{\partial J}{\partial t}(\xi, t) + \frac{\partial J}{\partial \xi}(\xi, t)\, f(\xi, u(t), t) \;\leq\; 0,$$

and, since $u(t)$ ranges over all of Ω as u varies, the inequality holds for every $u(t) \in \Omega$, with equality at the optimal control by (6.A.31). □

Proof of Theorem 6.4.1. We establish the necessary conditions under the standing hypothesis that the value function $J(\xi, t)$ of (6.4.6) is of class C^1 (and C^2 for the costate equation) on a neighborhood of $\{(x^*(t), t) : t \geq t_0\}$, the same hypothesis under which the Hamilton–Jacobi–Bellman equation of Theorem 6.A.3 was established. Set $\lambda^*(t) = (\partial J)/(\partial \xi)(x^*(t), t)$.

For the maximum condition, fix a point t of continuity of u^* and a control value $w \in \Omega$. For $h > 0$ let u_h agree with w on $[t, t+h]$ and follow an optimal continuation from the state reached at $t + h$, and let x_h be the induced state, so $x_h(t) = x^*(t)$. Optimality together with Lemma 6.A.3 gives

$$J(x^*(t), t) \;\geq\; \int_t^{t+h} F(x_h(s), w, s)\, ds + J(x_h(t+h), t+h).$$

Subtracting $J(x^*(t), t)$, dividing by h and letting $h \to 0^+$, and using $J \in C^1$ with $x_h' = f(x_h, w, s)$,

$$F(x^*(t), w, t) + \frac{\partial J}{\partial t}(x^*(t), t) + \frac{\partial J}{\partial \xi}(x^*(t), t)\, f(x^*(t), w, t) \;\leq\; 0.$$

Along the optimum, equality holds by (6.A.31), attained at $w = u^*(t)$. Writing $H(x,u,\lambda,t) = F(x,u,t) + \lambda\, f(x,u,t)$, for every $w \in \Omega$,

$$H\left(x^*(t), u^*(t), \lambda^*(t), t\right) \;\geq\; H\left(x^*(t), w, \lambda^*(t), t\right),$$

which is the maximum condition; interiority yields $\partial H/\partial u = 0$ at $u^*(t)$.

For the costate equation, the pointwise Hamilton–Jacobi–Bellman equation of Theorem 6.A.3 gives, for every (ξ, t) in the neighborhood,

$$0 = \max_{w\in\Omega} \left\{ F(\xi, w, t) + \frac{\partial J}{\partial t}(\xi, t) + \frac{\partial J}{\partial \xi}(\xi, t)\, f(\xi, w, t) \right\}.$$

Fixing $w = u^*(t)$, the function

$$\xi \;\longmapsto\; F(\xi, u^*(t), t) + \frac{\partial J}{\partial t}(\xi, t) + \frac{\partial J}{\partial \xi}(\xi, t)\, f(\xi, u^*(t), t)$$

is therefore ≤ 0 on the neighborhood, and it vanishes at $\xi = x^*(t)$ by (6.A.31). Hence $\xi = x^*(t)$ is an interior maximizer, and its first-order condition, computed with $J \in C^2$, reads

$$\frac{\partial F}{\partial x} + \frac{\partial^2 J}{\partial t\, \partial \xi} + \frac{\partial^2 J}{\partial \xi^2}\, f + \frac{\partial J}{\partial \xi}\, \frac{\partial f}{\partial x} \;=\; 0,$$

all evaluated along the optimum. Since $\lambda^*(t) = \frac{\partial J}{\partial \xi}(x^*(t), t)$, the chain rule gives $(\lambda^*)'(t) = (\partial^2 J)/(\partial t\, \partial \xi) + (\partial^2 J)/(\partial \xi^2)\, f$, whence

$$(\lambda^*)'(t) \;=\; -\left(\frac{\partial F}{\partial x} + \lambda^*(t)\, \frac{\partial f}{\partial x} \right) \;=\; -\frac{\partial H}{\partial x}\left(x^*(t), u^*(t), \lambda^*(t), t\right).$$

The state equation $(x^*)' = f(x^*, u^*, t)$ holds by admissibility. No terminal condition at infinity is produced; that is supplied separately by Theorem 6.4.2. The differentiability of J assumed here holds, for instance, under the concavity and interiority conditions of Benveniste and Scheinkman (1979); for the necessary conditions in the concave infinite-horizon setting, see also Araujo and Scheinkman (1983). $\square$

Proof of Theorem 6.4.2. For $t \geq T$, Equation (6.A.31) holds along the optimal pair:

$$F(x^*, u^*, t) + \frac{\partial J}{\partial t}(x^*(t), t) + \frac{\partial J}{\partial \xi}(x^*(t), t)\, f(x^*, u^*, t) = 0.$$

By the standard identification of the costate with the marginal value of the state, $\lambda^*(t) = (\partial J/\partial \xi)(x^*(t), t)$ (see Acemoglu (2009), eq. (7.43)). Substituting this identity and recalling that $H(x, u, \lambda, t) = F(x, u, t) + \lambda\, f(x, u, t)$, the first and third terms combine into the Hamiltonian, so that

$$\frac{\partial J}{\partial t}(x^*(t), t) + H\left(x^*(t), u^*(t), \lambda^*(t), t\right) = 0, \qquad t \geq T.$$

Letting $t \to \infty$ and using (6.4.7) yields (6.4.8). The Hamiltonian form of this condition, together with a proof not requiring differentiability of the value function, is due to Michel (1982). $\square$

Proof of Theorem 6.4.3. For the present-value form (6.4.13), apply Theorem 6.4.2 to the present-value Hamiltonian

$$H(x, u, \lambda, t) \;=\; e^{-\rho t} F(x, u) + \lambda\, f(x, u, t), \qquad \lambda(t) = e^{-\rho t}\psi(t).$$

Since $H = e^{-\rho t}\mathcal{H}$, (6.4.8) becomes (6.4.13). The strengthened form (6.4.15), obtained under the monotonicity and boundedness hypotheses (6.4.14) and the asymptotic-regime alternative, is technical; we refer to Acemoglu (2009), Theorem 7.13, for a complete derivation. □

Proof of Theorem 6.4.4. The argument parallels the finite-horizon Mangasarian–Arrow proof (Section 6.2); we work throughout in present-value form, setting $\lambda^*(t) = e^{-\rho t}\psi^*(t)$. By concavity of $x \mapsto M(x, \psi^*, t)$,

$$M(x, \psi^*, t) \;\le\; M(x^*, \psi^*, t) + \frac{\partial M}{\partial x}(x^*, \psi^*, t)\,(x - x^*). \qquad (6.A.32)$$

By the envelope theorem[36] applied to $M(x, \psi, t) = \max_{v \in \Omega} \mathcal{H}(x, v, \psi, t)$,

$$\frac{\partial M}{\partial x}(x^*, \psi^*, t) \;=\; \frac{\partial \mathcal{H}}{\partial x}(x^*, u^*, \psi^*, t),$$

and substituting the costate equation (6.4.12),

$$\frac{\partial M}{\partial x}(x^*, \psi^*, t) \;=\; \rho\,\psi^*(t) - (\psi^*)'(t). \qquad (6.A.33)$$

Multiplying (6.A.33) by $e^{-\rho t}$ and using $(\lambda^*)' = e^{-\rho t}\left((\psi^*)' - \rho\,\psi^*\right)$,

$$e^{-\rho t}\,\frac{\partial M}{\partial x}(x^*, \psi^*, t) \;=\; -(\lambda^*)'(t). \qquad (6.A.34)$$

For every admissible u and the optimal u^*, the definition of M and the maximum condition give

$$M(x, \psi^*, t) \;\ge\; \mathcal{H}(x, u, \psi^*, t), \qquad M(x^*, \psi^*, t) \;=\; \mathcal{H}(x^*, u^*, \psi^*, t).$$

[36] This is Danskin's theorem: if the maximizer u^* is unique, $x \mapsto M(x, \psi, t)$ is differentiable with derivative $\partial\mathcal{H}/\partial x$ at u^*, **why?** Concavity of M in x is all that the inequality (6.A.32) requires; the envelope equality (6.A.34) uses this differentiability, which we assume holds along the optimal path, as it does in all the applications below, where the maximizer is unique.

Subtracting these and using (6.A.32) and (6.A.34),

$$\begin{aligned} e^{-\rho t}\left[\mathcal{H}(x,u,\psi^*,t)-\mathcal{H}(x^*,u^*,\psi^*,t)\right] \;&\le\; e^{-\rho t}\,\frac{\partial M}{\partial x}(x^*,\psi^*,t)\,(x-x^*) \\ &=\; -(\lambda^*)'(t)\,(x-x^*). \end{aligned}$$

Since $\mathcal{H}(x,u,\psi^*,t) = F(x,u)+\psi^* f(x,u,t)$ and $\lambda^* = e^{-\rho t}\psi^*$, the left-hand side equals

$$e^{-\rho t}\left[F(x,u)-F(x^*,u^*)\right]+\lambda^*\left[f(x,u,t)-f(x^*,u^*,t)\right].$$

Using the state equations $f(x,u,t)-f(x^*,u^*,t)=x'-(x^*)'$,

$$\begin{aligned} e^{-\rho t}\left[F(x,u)-F(x^*,u^*)\right] \;&\le\; -(\lambda^*)'(t)\,(x-x^*)-\lambda^*(t)\,(x'-(x^*)') \\ &=\; -\frac{d}{dt}\left[\lambda^*(t)\,(x-x^*)\right]. \end{aligned}$$

Integrating over $[t_0,T]$ and using $x(t_0)=x^*(t_0)=x_0$, the boundary term at t_0 vanishes and

$$\int_{t_0}^{T} e^{-\rho t}\left[F(x,u)-F(x^*,u^*)\right]dt \;\le\; -\,\lambda^*(T)\,(x(T)-x^*(T))\,.$$

This holds for every $T \ge t_0$. If $\int_{t_0}^{\infty} e^{-\rho t}F(x,u)\,dt = -\infty$ the optimality inequality is immediate, so assume this integral is finite; letting $T\to\infty$, the left-hand side converges by (A3), so

$$\int_{t_0}^{\infty} e^{-\rho t}\left[F(x,u)-F(x^*,u^*)\right]dt \;\le\; -\limsup_{t\to\infty}\lambda^*(t)\,(x(t)-x^*(t))\,.$$

By (6.4.15) in present-value form, $\lim\limits_{t\to\infty}\lambda^*(t)\,x^*(t)=0$; hence

$$\begin{aligned} \limsup_{t\to\infty}\lambda^*(t)\,(x(t)-x^*(t)) &= \limsup_{t\to\infty}\lambda^*(t)\,x(t) \\ &\ge \liminf_{t\to\infty}\lambda^*(t)\,x(t) \\ &\ge 0 \end{aligned}$$

by hypothesis (6.4.16). The right-hand side is therefore nonpositive, and

$$\int_{t_0}^{\infty} e^{-\rho t} F(x,u)\,dt \;\le\; \int_{t_0}^{\infty} e^{-\rho t} F(x^*,u^*)\,dt,$$

which is the global-optimality claim. Strict concavity of M in x converts (6.A.32) into a strict inequality on $\{x \neq x^*\}$, yielding uniqueness of the optimal state trajectory. □

CONTINUITY OF THE OPTIMAL CONTROL

The result below upgrades the piecewise continuity of the optimal control to continuity. This is not a cosmetic point: the applications repeatedly differentiate the first-order condition in time, which presupposes a control without jumps.

Theorem 6.A.4. (Continuity of the optimal control). Under the hypotheses of Theorem 6.4.4, suppose further that $M(x,\psi,t)$ is strictly concave in x, $u \mapsto \mathcal{H}(x,u,\psi,t)$ is strictly concave for every (x,ψ,t), and Ω is compact. Then $u^* : [t_0,\infty[\to \Omega$ is continuous.

Proof. Strict concavity of $u \mapsto \mathcal{H}(x,u,\psi,t)$ on the compact set Ω implies that, for every (x,ψ,t), the maximizer

$$u^\dagger(x,\psi,t) \;=\; \arg\max_{u\in\Omega} \mathcal{H}(x,u,\psi,t)$$

is unique. We first show $u^\dagger$ is continuous. Fix $(\bar{x},\bar{\psi},\bar{t})$ and let $(x_n,\psi_n,t_n) \to (\bar{x},\bar{\psi},\bar{t})$. Set $u_n = u^\dagger(x_n,\psi_n,t_n) \in \Omega$. By compactness, every subsequence of $\{u_n\}$ admits a further subsequence converging to some $u^\circ \in \Omega$. Along it, $\mathcal{H}(x_n,u_n,\psi_n,t_n) \ge \mathcal{H}(x_n,u,\psi_n,t_n)$ for every $u \in \Omega$; passing to the limit, using continuity of $\mathcal{H} = F + \psi f$ (which follows from $F, f \in C^1$), gives $\mathcal{H}(\bar{x}, u^\circ, \bar{\psi}, \bar{t}) \geq \mathcal{H}(\bar{x}, u, \bar{\psi}, \bar{t})$ for every $u \in \Omega$. Uniqueness forces $u^\circ = u^\dagger(\bar{x}, \bar{\psi}, \bar{t})$. Hence every subsequential limit of $\{u_n\}$ equals $u^\dagger(\bar{x}, \bar{\psi}, \bar{t})$, and since $\{u_n\}$ lies in the compact Ω, the whole sequence converges to that limit.

The trajectories x^* and ψ^* are continuous: by Theorem 6.4.1, λ^* is absolutely continuous, so $\psi^* = e^{\rho t}\lambda^*$ is continuous; and the representation $x^*(t) = x_0 + \int_{t_0}^t f(x^*, u^*, s)\, ds$ ensures x^* is continuous.

Define $\varphi(t) = u^\dagger(x^*(t), \psi^*(t), t)$. By the previous two paragraphs, φ is continuous on $[t_0, \infty)$ as a composition of continuous functions; this construction requires no differentiability of u^*. By the Maximum Principle and uniqueness of the maximizer, $u^*(t)$ and $\varphi(t)$ both equal the unique maximizer of $\mathcal{H}(x^*(t), \cdot, \psi^*(t), t)$ at every t where u^* is continuous; hence $u^*(t) = \varphi(t)$ on this set, which we denote $\mathcal{D}$.

We claim $\mathcal{D}$ is dense in $[t_0, \infty[$. Since u^* is piecewise continuous, in each compact interval its discontinuities are finite in number and therefore isolated; every point of $[t_0, \infty)$ thus admits points of $\mathcal{D}$ arbitrarily close to it. Now fix any $\hat{t} \in [t_0, \infty[$. Piecewise continuity of u^* guarantees that the one-sided limits $u^*(\hat{t}^\pm)$ exist. By density of $\mathcal{D}$, choose sequences in $\mathcal{D}$ approaching $\hat{t}$ from each side; along them $u^* = \varphi$, and continuity of φ gives $u^*(\hat{t}^-) = u^*(\hat{t}^+) = \varphi(\hat{t})$. The two one-sided limits coincide, so u^* has no jump at $\hat{t}$ and takes the value $\varphi(\hat{t})$ there. Since $\hat{t}$ was arbitrary, $u^* = \varphi$ throughout $[t_0, \infty)$, and u^* inherits the continuity of φ. $\square$