

Dynamic Demand and Seasonal Markdown Pricing

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Abstract

We study demand and seasonal pricing for two school-trouser sizes in fourteen Chilean stores during the 2015 back-to-school campaign. Weekly posted prices are reconstructed from transaction payments. Consumers choose to buy, wait, or exit, with permanent normal heterogeneity in price sensitivity and price forecasts calibrated to matched 2014 products. EM integrates over customer types and nonbuyers' latent exit times. The two SKUs exhibit positive estimated dispersion under conventional Wald tests. A finite-horizon retailer Bellman equation tracks the expected population remaining by type and uses the same consumer forecasts. Optimal prices are conditional on historical beliefs, sufficient availability, and the specified markdown grid.

Keywords: seasonal demand; price expectations; heterogeneous preferences; EM estimation; dynamic pricing.

1 Introduction

Seasonal prices influence both whether consumers buy and when they buy. A customer compares the current offer with expected future offers and the option to leave. Price expectations therefore enter both demand estimation and the retailer's intertemporal decision (Aviv & Pazgal, 2008; Nair, 2007; Su, 2007).

We study two school-trouser sizes sold in fourteen Chilean stores during the twelve-week 2015 back-to-school campaign. An explicit algorithm reconstructs weekly posted prices from transaction payments. Matched 2014 products supply a historical price-transition rule: consumers observe today's price and forecast future price states instead of knowing the future realized path.

Customers choose to buy, wait, or exit, with permanent normal heterogeneity in price sensitivity. We derive the Bellman equation, the likelihood of first-purchase histories, and the Bayesian reconstruction of unobserved types and exits used by EM. The waiting-flow indicator begins in week 8 for both SKUs. Both exhibit positive estimated dispersion under conventional Wald tests.

The retailer then solves a finite-horizon pricing problem with the same estimated demand and consumer forecasts. Its state tracks the remaining population by type and the previous price. Today's price affects revenue and the composition of customers available tomorrow. The exercise holds historical beliefs fixed and evaluates expected revenue over the admissible markdown grid; it

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does not impose that consumers anticipate the optimized schedule or solve a rational-expectations equilibrium.

2 Data and posted prices

2.1 The 2015 campaign

The Chilean retailer’s 2015 back-to-school extract contains 283,644 transaction records, 30,147 customer identifiers, and 14 stores. Records identify the customer, store, SKU, payment, and campaign week (codes 0–13). We study two Club Maxx boys’ gray trouser SKUs: 1560649 (T14–16) and 1560650 (T38–50). Their list prices are CLP 6,990 and 7,990.

Figures 1 and 2 display positive-payment transaction lines by week, in each store and across stores. These counts include repeat purchases; they do not infer unit quantities. The all-item figure includes the generic code 0 and all 22 individually identified products. Descriptive figures cover weeks 0–13; estimation covers weeks 1–12.

For each SKU, the sample starts from every customer observed in 2015. The outcome is the first positive-payment purchase in weeks 1–12, or no purchase. Week-0 purchasers are excluded; first purchases in week 13 count as no purchase within the horizon. The two trouser samples contain 30,137 and 30,141 customers, with 2,450 and 2,662 first purchasers. Samples overlap across products.

A buyer’s store is the first-purchase store; same-week ties favor more positive SKU lines, then the lowest store code. A nonbuyer’s store is the most frequent store across positive-payment lines, breaking ties by spending and store code. If there are no positive payments, all lines determine the most frequent store. All sampled customers enter in week 1; waiting and exit are unobserved.

2.2 Posted-price reconstruction

The list price P_j^{list} comes from the retailer’s plan and category catalog. A posted price P_{jlt}^{post} is the unit price after the seasonal markdown, before an additional payment discount. A recorded payment y_r can cover several units. For a candidate markdown η , generate a candidate unit price P and test

$$P = P_j^{\text{list}}(1 - \eta), \quad y_r \simeq kaP, \quad k \in \{1, \dots, 6\}, \quad a \in \{1, 0.95\}. \quad (1)$$

Here $a = 1$ means no additional discount and $a = 0.95$ means an additional 5% off. For example, two units of SKU 1560649 with a 20% markdown and the additional discount generate $y_r = 2 \times 0.95 \times (0.8 \times 6,990) = 10,624.8$ pesos. The posted unit price is CLP 5,592.

The algorithm pools the four trouser SKUs (1560647–1560650) within each store-week; the two additional sizes are used only for price reconstruction. Tables 1 and 2 give each focal product’s candidate prices and activation weeks.

1. Retain compatible explanations. For each positive-payment line, retain candidates satisfying

$$e_r(P, k, a) = \frac{|y_r/(ka) - P|}{P} \leq 0.005 \quad (2)$$

and within 10^{-4} of that line's smallest error. A line with no compatible explanation is unused for prices but may still define a purchase.

- 2. Resolve ambiguous lines.** Use uniquely identified percentage-grid prices to obtain the modal relative price within category and store-week, requiring three lines; otherwise use that category's chain-wide weekly mode. Select each ambiguous line's explanation closest to this preliminary mode. Ties favor no additional discount, then fewer units; modal ties favor the higher price.
- 3. Select a cell's modal price.** Use an own-SKU earlier-list or clearance mode when at least three compatible own-SKU lines exist and at least half support it. Otherwise use the category's store-week percentage-grid mode, requiring three lines; failing that, use its chain-week mode. Denote the selected relative price by $\tilde{p}_{j\ell t}$ and the size of its supporting group, at least one, by $w_{j\ell t}$.
- 4. Construct the weekly posted path.** Treat markdowns as permanent. Weighted isotonic regression, computed by pooling adjacent violating blocks, gives

$$\bar{p}_{j\ell,1:12} = \arg \min_{x_1 \geq \dots \geq x_{12}} \sum_{t=1}^{12} w_{j\ell t} (x_t - \tilde{p}_{j\ell t})^2. \quad (3)$$

Round each fitted value to the nearest candidate relative price for its week, with lower-price ties, obtaining $\check{p}_{j\ell t}$. Then set

$$p_{j\ell t} = \min_{u \leq t} \check{p}_{j\ell u}, \quad P_{j\ell t}^{\text{post}} = P_j^{\text{list}} p_{j\ell t}. \quad (4)$$

The resulting 14×12 relative-price matrix is the estimation input. Weeks 0 and 13 are outside this adjustment and the model horizon.

Figures 3 and 4 plot the reconstructed prices. The category pooling and nonincreasing-path restriction are maintained reconstruction assumptions; they do not convert payments into directly observed shelf prices.

Table 1: List price and admissible discounts: SKU 1560649, trousers T14–16

Price level	Discount (%)	Price (CLP)	Candidate weeks
List price	0	6,990	0–13
Markdown	10	6,291	0–13
Markdown	20	5,592	0–13
Markdown	30	4,893	0–13
Markdown	40	4,194	0–13
Markdown	50	3,495	9–13
Earlier list price	14.31	5,990	0–1
Clearance	71.53	1,990	11–13

Table 2: List price and admissible discounts: SKU 1560650, trousers T38–50

Price level	Discount (%)	Price (CLP)	Candidate weeks
List price	0	7,990	0–13
Markdown	10	7,191	0–13
Markdown	20	6,392	0–13
Markdown	30	5,593	0–13
Markdown	40	4,794	0–13
Markdown	50	3,995	9–13
Earlier list price	12.52	6,990	0–1
Clearance	75.09	1,990	11–13

Notes: Discounts are relative to each SKU’s list price. Candidate weeks are campaign codes. Earlier lists come from the category catalog; clearance candidates are CLP 1,990. The additional 5% payment discount is allowed in reconstruction and is not part of the posted price.

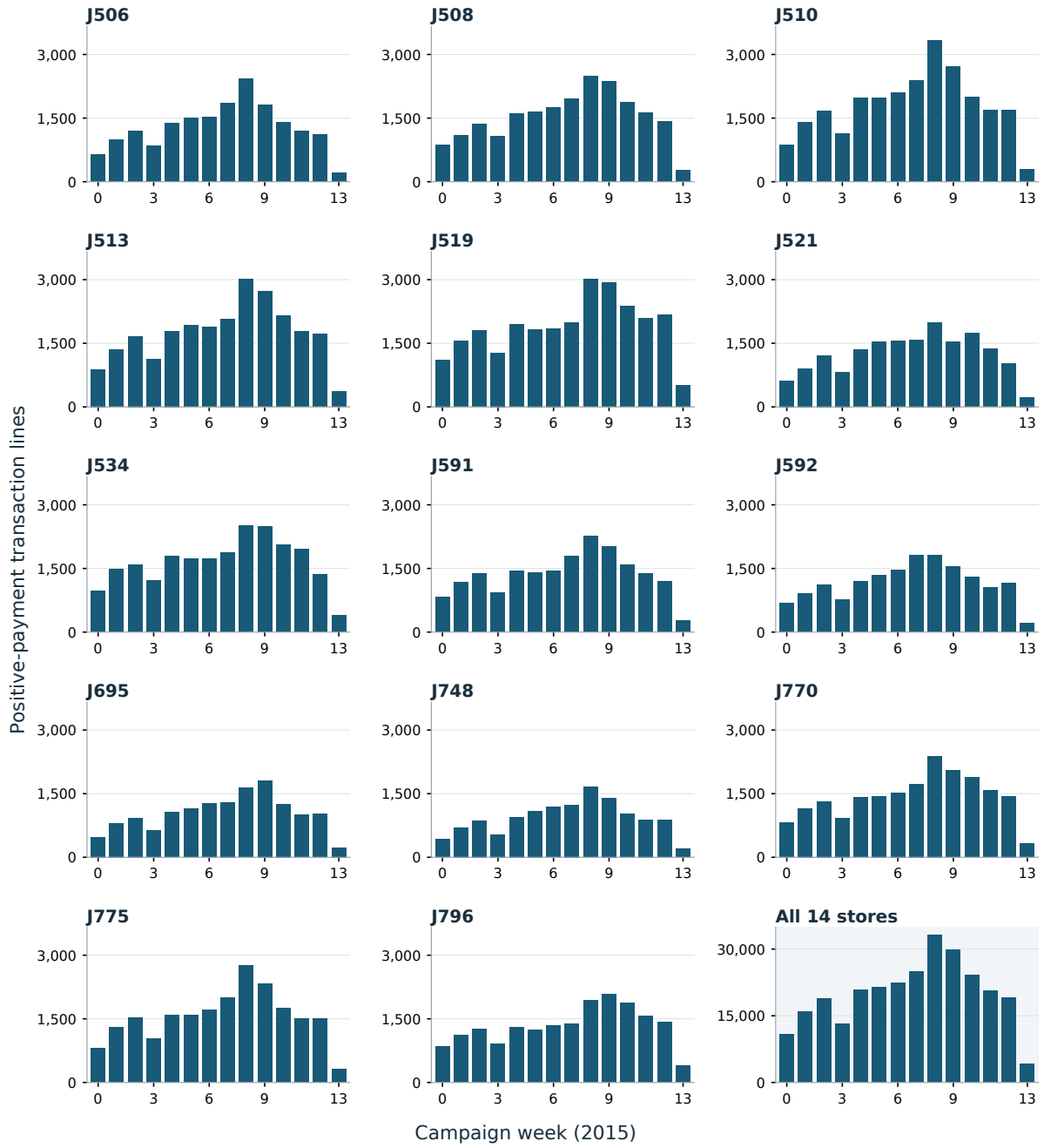


Figure 1: Weekly purchases of all items, 2015. Bars count positive-payment transaction lines. The first 14 panels share a vertical scale; the shaded panel aggregates all stores and has its own scale.

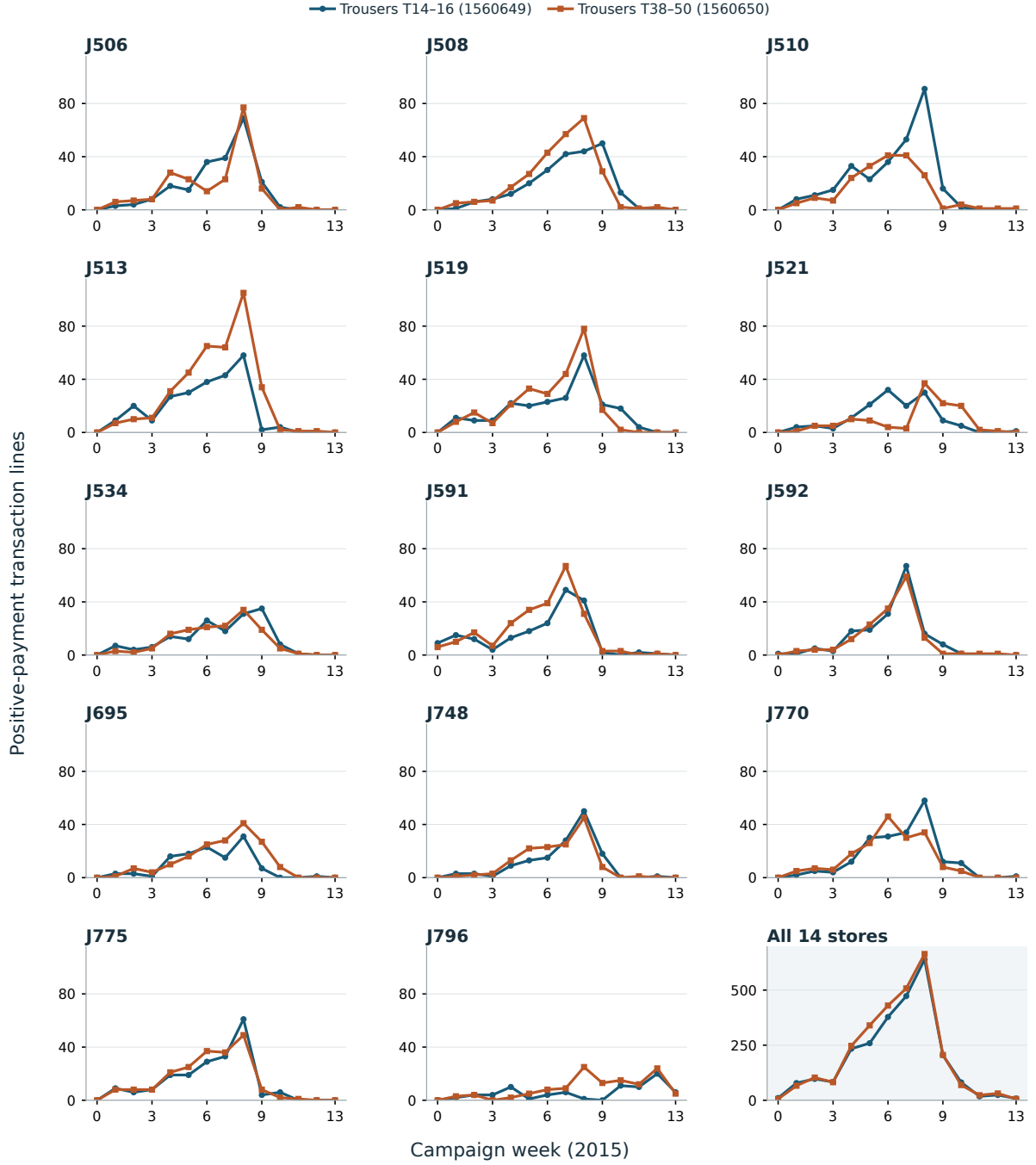


Figure 2: Weekly purchases of the two focal SKUs, 2015. Lines count positive-payment transaction records, including repeat purchases. The first 14 panels share a vertical scale; the shaded panel aggregates all stores and has its own scale.

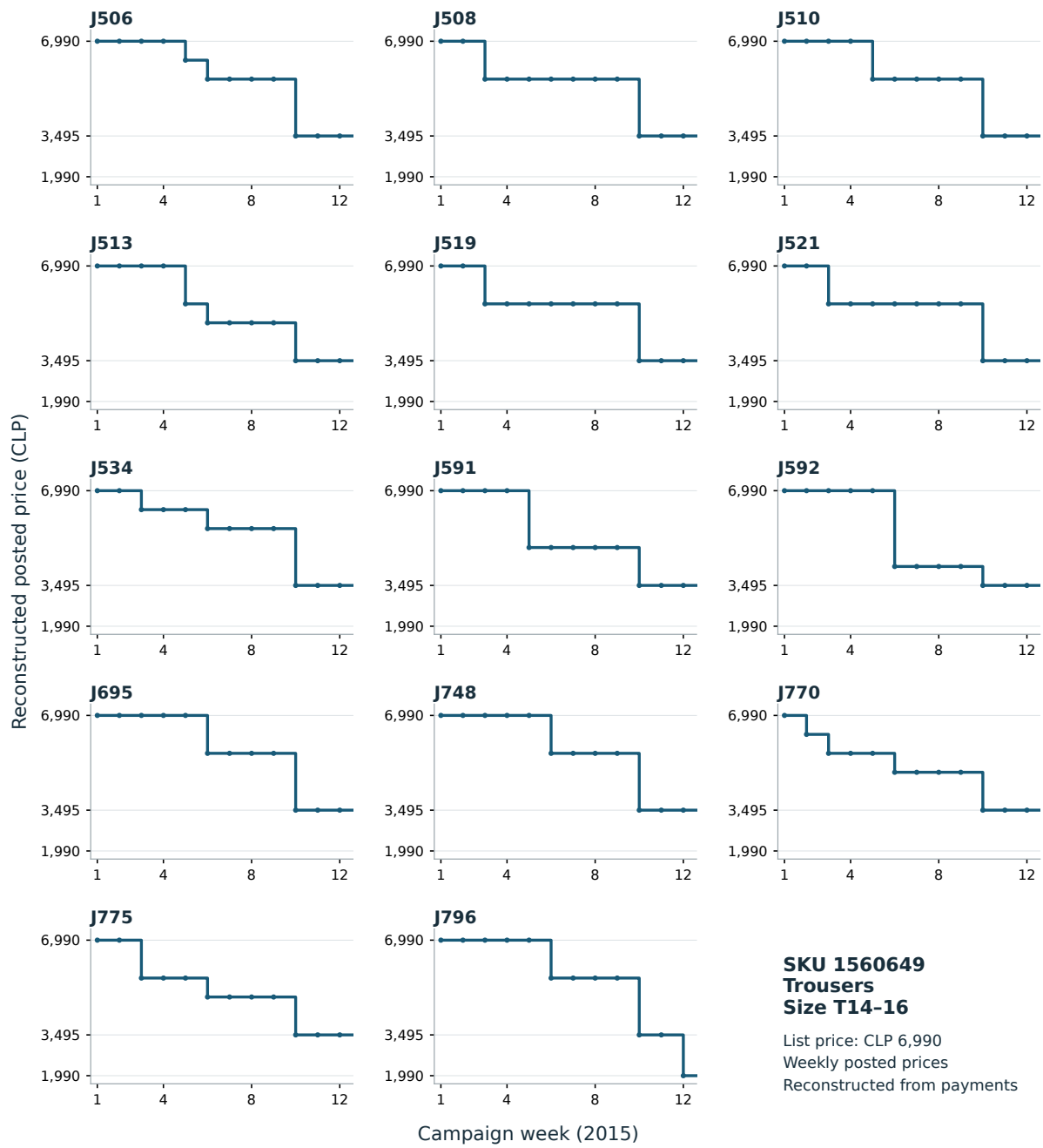


Figure 3: Reconstructed posted prices for SKU 1560649, trousers T14–16, by store in 2015. Prices are in CLP; all panels share the same axes. Relative prices divide these values by the CLP 6,990 list price.

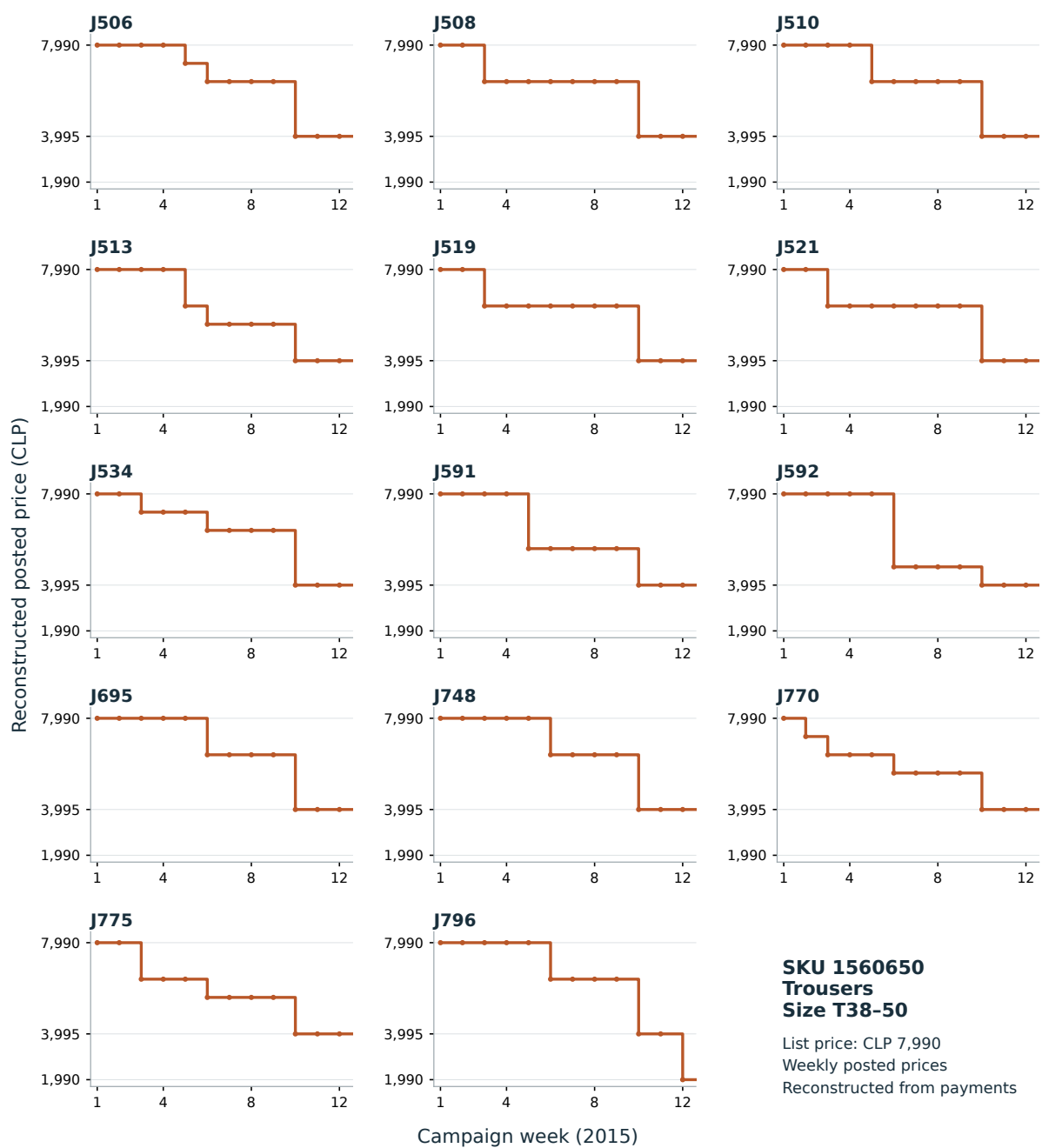


Figure 4: Reconstructed posted prices for SKU 1560650, trousers T38–50, by store in 2015. Prices are in CLP; all panels share the same axes. Relative prices divide these values by the CLP 7,990 list price.

3 Dynamic demand, price expectations, and EM

3.1 Price forecasts from prior-season information

Consumers observe the current posted price, but do not know the future realized price path. We specify a historical forecasting rule before estimating demand. The 2014 description matches are SKU 1356212 for 1560649 and 1356213 for 1560650. These earlier observations construct beliefs; only 2015 purchase histories enter the demand likelihood.

For each matched product, normalize 2014 payments by their modal amount in weeks 1–2 across the same fourteen stores. In each store-week, retain positive payments no greater than this anchor; a cell requires three lines and two observations at its mode, with higher-price ties. The modal payment divided by the anchor is a historical price proxy. Missing cells remain missing; quantities and payment discounts are not separately observed in these proxies.

Let $b(p) \in \{0, 1, 2\}$ classify relative prices as low ($p \leq 0.5$), intermediate ($0.5 < p < 0.9$), or high ($p \geq 0.9$). Let g_{js} be the mean of valid 2014 proxies in state s for product j . Pool adjacent observed 2014 cells across stores, using transition origins 2–11. With calendar phase $h(t) = 1$ for $t \leq 5$ and $h(t) = 2$ thereafter, the historical transition rule is

$$\widehat{K}_{jh}(s, s') = \frac{C_{jhs s'}^{2014} + 1/3}{\sum_r C_{jhs r}^{2014} + 1}. \quad (5)$$

Here $C_{jhs s'}^{2014}$ counts transitions from state s to s' in phase h . The fixed unit pseudocount regularizes sparse rows; the state definitions, phases, and smoothing rule are not selected using the 2015 demand fit.

At week t , the information set $\mathcal{I}_{\ell t}$ contains the calendar, current price $p_{\ell t}$, and the historical rule. Past prices enter through that rule and the current state, a first-order Markov restriction. The forecast is

$$\mathbb{E}_j[p_{\ell, t+1} \mid \mathcal{I}_{\ell t}] = \sum_{s'} \widehat{K}_{j, h(t)}(b(p_{\ell t}), s') g_{js'}. \quad (6)$$

Multiweek forecasts multiply the successive transition matrices. Beliefs remain fixed during 2015; observing a new current price changes its state and hence the forecast. Neither future 2015 prices nor future sales update today’s rule.

Table 3: Historical price forecasts: training support and 2015 validation

SKU	2014 pairs	2015 pairs	Historical MAE	Persistence MAE
1560649	94	87	0.107	0.081
1560650	100	85	0.103	0.078

MAE is the absolute error of the one-week-ahead relative-payment proxy forecast. Both rules use the same complete 2015 cell pairs, at origins 2–11. Kernel estimation and state-price representatives use only 2014. These are proxy forecasts, not directly observed customer expectations.

The historical rule has higher one-week absolute error than persistence for both SKUs (Table 3). It is a specified forecasting mechanism, not an elicited or uniquely identified consumer belief. Validation uses payment proxies; the demand model observes reconstructed current posted

prices, whose retrospective smoothing uses the full 2015 season. Inference below conditions on both constructions.

3.2 Choices, timing, and the Bellman equation

We estimate the two SKUs separately, suppressing j when unambiguous. A customer enters week 1 and chooses buy (B), wait (W), or exit (O) while active. Buy and exit are absorbing; waiting is unavailable at $T = 12$. The permanent price coefficient is $\beta_i = \bar{\beta} + \sigma_\beta z_i$, $z_i \sim N(0, 1)$. The customer knows her type, the researcher does not, and its initial distribution is common across stores. Only β_i is heterogeneous; $\rho = 0.9$ is fixed. At current relative price p ,

$$\begin{aligned} u_t^B(z, p) &= \alpha - (\bar{\beta} + \sigma_\beta z)p + \delta_{12} \mathbf{1}\{t \leq 2\}, & u_t^O &= \gamma, \\ u_t^W &= d \mathbf{1}\{t \geq 8\}, & t < T. \end{aligned} \quad (7)$$

The waiting-flow shift applies in weeks 8–11; waiting is unavailable in week 12. The baseline waiting payoff is zero. Independent, mean-zero unit-scale extreme-value shocks ε_{it}^a are observed before the current choice; future shocks and future prices are unknown.

Write D_i for the first buy or exit week. Conditional on activity, the customer chooses a policy adapted to her information to maximize

$$\mathcal{V}_t(z, p, \varepsilon_t) = \max_{\text{decision policies}} \mathbb{E}_{\widehat{K}, \varepsilon} \left[\sum_{s=t}^{D_i} \rho^{s-t} (u_s^a(z, p_s) + \varepsilon_s^a) \mid z, \mathcal{I}_{\ell t}, \varepsilon_t \right]. \quad (8)$$

The expectation integrates future price states and future choice shocks. Let $V_t(z, p) = \mathbb{E}_{\varepsilon_t}[\mathcal{V}_t(z, p, \varepsilon_t)]$ and define

$$\mathcal{C}_t(z, p) = \sum_{s'} \widehat{K}_{h(t)}(b(p), s') V_{t+1}(z, g_{s'}). \quad (9)$$

Separating the first payoff from the remaining sum gives

$$\mathcal{V}_t(z, p, \varepsilon_t) = \max\{u_t^B(z, p) + \varepsilon_t^B, \gamma + \varepsilon_t^O, u_t^W + \rho \mathcal{C}_t(z, p) + \varepsilon_t^W\}, \quad t < T. \quad (10)$$

Only waiting has a continuation payoff. The choice-specific values are

$$v_t^B = u_t^B(z, p), \quad v_t^O = \gamma, \quad v_t^W = u_t^W + \rho \mathcal{C}_t(z, p), \quad t < T. \quad (11)$$

These are standard dynamic discrete-choice Bellman equations with an explicit price transition rule (Rust, 1987; Train, 2009); forward-looking price expectations also underlie intertemporal pricing models (Nair, 2007).

3.3 From values to choice probabilities

For the mean-zero extreme-value distribution, $F_\varepsilon(e) = \exp\{-e^{-(e+\kappa)}\}$, where κ is Euler's constant. Independence gives, for $M = \max_{a \in \mathcal{A}_t} (v_t^a + \varepsilon_t^a)$,

$$\mathbb{P}(M \leq m \mid z, p) = \prod_{a \in \mathcal{A}_t} F_\varepsilon(m - v_t^a) = \exp\left\{-e^{-(m+\kappa)} \sum_a e^{v_t^a}\right\}. \quad (12)$$

Its mean is $\log \sum_a e^{v_t^a}$: the location shift $-\kappa$ cancels the extreme-value mean. Therefore

$$\begin{aligned} V_T(z, p) &= \log(e^{u_T^B(z, p)} + e^\gamma), \\ V_t(z, p) &= \log(e^{u_t^B(z, p)} + e^\gamma + e^{u_t^W + \rho \mathcal{C}_t(z, p)}), \quad t < T. \end{aligned} \quad (13)$$

Backward induction first solves these values at the three future-price states; (9) then evaluates waiting at each observed current price. It integrates the future value, rather than substituting a mean price inside a nonlinear value function.

With continuous shocks, the derivative of the expected maximum with respect to a choice-specific value is its choice probability (Rust, 1987, Theorem 1). Thus, with $\mathcal{A}_t = \{B, W, O\}$ for $t < T$ and $\mathcal{A}_T = \{B, O\}$,

$$\pi_{\ell t}^a(z) = \pi_t^a(z, p_{\ell t}; \widehat{K}) = \frac{\partial V_t}{\partial v_t^a} = \frac{e^{v_t^a(z, p_{\ell t})}}{\sum_{b \in \mathcal{A}_t} e^{v_t^b(z, p_{\ell t})}}. \quad (14)$$

This is the dynamic logit identity (Train, 2009, Sections 3.1 and 3.5). At T , $\pi^W = 0$. Historical prices affect current demand through the expected continuation value, not through knowledge of the realized future path.

3.4 Purchase histories and latent exit

A first purchase in week t requires waiting in every preceding week. Let $S_{\ell t}(z)$ be the probability of still being active at the start of t , conditional on type. Starting with $S_{\ell 1}(z) = 1$, sequential conditioning gives

$$S_{\ell, t+1}(z) = S_{\ell t}(z) \pi_{\ell t}^W(z), \quad S_{\ell t}(z) = \prod_{s < t} \pi_{\ell s}^W(z). \quad (15)$$

Multiplying survival by the next absorbing choice gives the probabilities of the two possible completed histories:

$$H_{\ell t}^B(z) = S_{\ell t}(z) \pi_{\ell t}^B(z), \quad H_{\ell t}^O(z) = S_{\ell t}(z) \pi_{\ell t}^O(z). \quad (16)$$

For example, $H_{\ell 3}^B = \pi_{\ell 1}^W \pi_{\ell 2}^W \pi_{\ell 3}^B$ represents waiting twice and then buying. A customer without a purchase can instead have waited until any week t and exited then. Her record does not reveal that exit week; it does not imply that she waited throughout the season.

Since $\pi^B + \pi^O = 1 - \pi^W$, $H_{\ell t}^B(z) + H_{\ell t}^O(z) = S_{\ell t}(z) - S_{\ell, t+1}(z)$. Summing over weeks telescopes to one because $S_{\ell 1} = 1$ and $S_{\ell, T+1} = 0$. Thus buying in weeks $1, \dots, T$ and exiting in those weeks are mutually exclusive histories that exhaust the model's outcomes.

3.5 Complete and observed likelihoods

Let $c_i = t > 0$ denote a first purchase in week t , and $c_i = 0$ denote no purchase. The latent objects are the permanent type z_i and, for nonbuyers, the exit week D_i ; for buyers, $D_i = c_i$ is observed. Use k to index weeks within an individual history and t for grouped store-week counts. Define

$$w_{ik} = \mathbf{1}\{k \leq D_i\}, \quad k = 1, \dots, T+1, \quad w_{i1} = 1, \quad w_{i, T+1} = 0. \quad (17)$$

Thus w_{ik} indicates activity at the start of week k . With $b_i = \mathbf{1}\{c_i > 0\}$ and $b_{ik} = \mathbf{1}\{c_i = k\}$,

$$\mathbf{1}\{\text{wait at } k\} = w_{i,k+1}, \quad o_{ik} = \mathbf{1}\{\text{exit at } k\} = (1 - b_i)(w_{ik} - w_{i,k+1}). \quad (18)$$

The exit indicator o_{ik} is therefore a change in activity for a nonbuyer. For buyers, $w_{ik} = \mathbf{1}\{k \leq c_i\}$ is determined by the purchase week. For nonbuyers, the intermediate activity indicators are unobserved: a weekly zero can reflect waiting, exiting that week, or having exited earlier.

If type and activity were observed, independent weekly shocks would give the complete individual likelihood

$$\begin{aligned} L_i^{\text{complete}}(\theta) &= \phi(z_i) \prod_{k=1}^{T-1} \left[\pi_{\ell(i)k}^W(z_i; \theta) \right]^{w_{i,k+1}} \\ &\times \prod_{k=1}^T \left[\pi_{\ell(i)k}^B(z_i; \theta) \right]^{b_{ik}} \left[\pi_{\ell(i)k}^O(z_i; \theta) \right]^{(1-b_i)(w_{ik} - w_{i,k+1})}. \end{aligned} \quad (19)$$

Here $\theta = (\alpha, \bar{\beta}, \sigma_\beta, \delta_{12}, d, \gamma)$ and ϕ is the standard-normal density. The waiting product stops before T , when waiting is unavailable. The expression equals $\phi(z_i)H_{\ell(i),c_i}^B(z_i)$ for a buyer and $\phi(z_i)H_{\ell(i),D_i}^O(z_i)$ for a nonbuyer.

To obtain the likelihood of the observed record, integrate over z_i and, for a nonbuyer, sum over every possible exit week. Hence

$$P_{\ell t}(\theta) = \int H_{\ell t}^B(z; \theta) \phi(z) dz, \quad t \geq 1, \quad P_{\ell 0}(\theta) = \int \left[\sum_{t=1}^T H_{\ell t}^O(z; \theta) \right] \phi(z) dz. \quad (20)$$

All history probabilities condition on the observed current-price sequence and the fixed historical kernel. The individual observed likelihood is $L_i(\theta) = P_{\ell(i),c_i}(\theta)$. The type is integrated once for the entire history because it is permanent. Multiplying separately integrated weekly choice probabilities would instead allow the type to change across weeks. For nonbuyers, summing over exits also occurs before taking the logarithm.

Customers in the same store with the same observed outcome have the same likelihood contribution. Writing $n_{\ell c}$ for their count therefore gives

$$L(\theta) = \prod_i P_{\ell(i),c_i}(\theta) = \prod_{\ell} \prod_{c=0}^T P_{\ell c}(\theta)^{n_{\ell c}}, \quad \ell(\theta) = \sum_{\ell} \sum_{c=0}^T n_{\ell c} \log P_{\ell c}(\theta). \quad (21)$$

The grouped multinomial likelihood differs only by a parameter-independent combinatorial factor. Every customer enters once, including nonbuyers.

3.6 Bayesian reconstruction and EM

Bayes' rule supplies probabilities for the missing activity histories. At iteration m , a nonbuyer's joint posterior density of type and exit week is

$$\underbrace{f(z_i = z, D_i = s \mid c_i = 0, \ell(i); \theta^{(m)})}_{\text{posterior}} = \frac{\overbrace{\phi(z) H_{\ell(i)s}^O(z; \theta^{(m)})}^{\text{joint density of type, exit in } s, \text{ and no purchase}}}{P_{\ell(i)0}(\theta^{(m)})}. \quad (22)$$

The numerator describes waiting through $s - 1$ and exiting at s with type z ; the denominator sums and integrates all histories consistent with no purchase. Integrating out type gives the posterior exit probability

$$\psi_{is}^{(m)} = \mathbb{P}(D_i = s \mid c_i = 0, \ell(i); \theta^{(m)}) = \frac{\int \phi(z) H_{\ell(i)s}^O(z; \theta^{(m)}) dz}{P_{\ell(i)0}(\theta^{(m)})}, \quad \sum_{s=1}^T \psi_{is}^{(m)} = 1. \quad (23)$$

A nonbuyer is active at k exactly when her exit occurs at k or later. Consequently, the reconstructed activity and action weights are

$$\begin{aligned} \bar{w}_{ik}^{(m)} &= \mathbb{E}[w_{ik} \mid c_i = 0, \ell(i); \theta^{(m)}] = \sum_{s=k}^T \psi_{is}^{(m)}, \\ \mathbb{E}[\mathbf{1}\{\text{wait at } k\} \mid c_i = 0, \ell(i); \theta^{(m)}] &= \bar{w}_{i,k+1}^{(m)}, \\ \mathbb{E}[o_{ik} \mid c_i = 0, \ell(i); \theta^{(m)}] &= \bar{w}_{ik}^{(m)} - \bar{w}_{i,k+1}^{(m)} = \psi_{ik}^{(m)}. \end{aligned} \quad (24)$$

These are posterior weights, rather than observed choices. EM uses them to form the expected complete log likelihood,

$$Q(\theta \mid \theta^{(m)}) = \sum_i \mathbb{E}_{\theta^{(m)}} \left[\log L_i^{\text{complete}}(\theta) \mid c_i, \ell(i) \right]. \quad (25)$$

The observed likelihood remains (21). With heterogeneous β_i , type and exit time have a joint posterior: we retain that joint distribution when weighting type-specific log probabilities in Q .

We approximate the normal integral with 121 equally spaced nodes z_q on $[-6, 6]$ and weights $\omega_q = \phi(z_q) / \sum_r \phi(z_r)$. The corresponding price coefficients are $\beta_q(\theta) = \bar{\beta} + \sigma_\beta z_q$. Let $P_{\ell c}^{(Q)}(\theta)$ denote (20) with integrals replaced by these weighted sums. The weights and standardized nodes stay fixed during estimation; the coefficients $\beta_q(\theta)$ move with the mean and dispersion.

In the E-step at $\theta^{(m)}$, Bayes' rule gives the type posterior for a buyer in week t , and the joint type-exit-week posterior for a nonbuyer:

$$r_{q\ell t}^B = \frac{\omega_q H_{\ell t}^B(z_q; \theta^{(m)})}{P_{\ell t}^{(Q)}(\theta^{(m)})}, \quad r_{q\ell t}^O = \frac{\omega_q H_{\ell t}^O(z_q; \theta^{(m)})}{P_{\ell 0}^{(Q)}(\theta^{(m)})}. \quad (26)$$

For buyers, $\sum_q r_{q\ell t}^B = 1$ for each purchase week. For nonbuyers, $\sum_{q,t} r_{q\ell t}^O = 1$ jointly over type and exit week. Thus $\sum_t r_{q\ell t}^O$ is their type posterior, whereas $\sum_q r_{q\ell t}^O$ is their exit-week posterior. The two groups need not have the same posterior type distribution. In particular, $\psi_{is}^{(m)} \simeq \sum_q r_{q\ell(i)s}^O$. Keeping type as well as activity, the posterior weight of type q and activity at k is

$$\bar{w}_{iqk}^{(m)} = \begin{cases} r_{q\ell(i),c_i}^B \mathbf{1}\{k \leq c_i\}, & c_i > 0, \\ \sum_{s=k}^T r_{q\ell(i)s}^O, & c_i = 0. \end{cases} \quad (27)$$

The expected waiting count below is $m_{q\ell t}^W = \sum_{i:\ell(i)=\ell} \bar{w}_{iq,t+1}^{(m)}$; the exit count is $m_{q\ell t}^O = \sum_{i:\ell(i)=\ell, c_i=0} (\bar{w}_{iq,t}^{(m)} - \bar{w}_{iq,t+1}^{(m)})$. Thus the grouped EM objective directly aggregates the Bayesian activity weights.

Multiplying these posteriors by the observed counts gives expected numbers of purchases and exits by type. Expected waiting counts follow because a customer absorbed after t must have waited at t :

$$m_{q\ell t}^B = n_{\ell t} r_{q\ell t}^B, \quad m_{q\ell t}^O = n_{\ell 0} r_{q\ell t}^O, \quad m_{q\ell t}^W = \sum_{s>t} (m_{q\ell s}^B + m_{q\ell s}^O), \quad t < T. \quad (28)$$

For the quadrature model, the complete contribution replaces the density $\phi(z_i)$ in (19) with the mass ω_q . Taking the conditional expectation of its logarithm gives

$$\begin{aligned} Q(\theta \mid \theta^{(m)}) &= C^{(m)} + \sum_{q,\ell,t} \left[m_{q\ell t}^B \log H_{\ell t}^B(z_q; \theta) + m_{q\ell t}^O \log H_{\ell t}^O(z_q; \theta) \right] \\ &= C^{(m)} + \sum_{q,\ell,t} \left[m_{q\ell t}^B \log \pi_{\ell t}^B(z_q; \theta) + m_{q\ell t}^O \log \pi_{\ell t}^O(z_q; \theta) \right] + \sum_{q,\ell,t < T} m_{q\ell t}^W \log \pi_{\ell t}^W(z_q; \theta), \end{aligned} \quad (29)$$

where $C^{(m)} = \sum_{q,\ell,t} (m_{q\ell t}^B + m_{q\ell t}^O) \log \omega_q$ is constant in the trial parameters. The second line follows by expanding each history into its waiting and absorbing choices. The expected counts remain fixed throughout the M-step.

The M-step numerically maximizes $Q(\theta \mid \theta^{(m)})$ over θ with $\sigma_\beta \geq 0$, accepting an update only when Q does not decrease. We solve the Bellman recursion again at each trial parameter vector. In particular, changing $\bar{\beta}$ or σ_β changes every type-specific purchase value and its continuation values: $\partial \beta_q / \partial \bar{\beta} = 1$ and $\partial \beta_q / \partial \sigma_\beta = z_q$. There are no free type masses to update separately.

For an ordinary EM step, Jensen's inequality yields $\ell_Q(\theta^{(m+1)}) - \ell_Q(\theta^{(m)}) \geq Q(\theta^{(m+1)} \mid \theta^{(m)}) - Q(\theta^{(m)} \mid \theta^{(m)}) \geq 0$, where ℓ_Q is the quadrature likelihood. We alternate these E- and M-steps, using fixed-point acceleration only when it preserves likelihood ascent. Dispersed starts and an exact zero-dispersion EM candidate, the likelihood score, a finer 321-node grid, and the probability Jacobian check convergence and local identification. Standard errors use the inverse observed Hessian of the observed likelihood, rather than the curvature of the complete-data objective Q .

Remark (identification). Magnac and Thesmar (2002, Proposition 2) show that discounting and unrestricted utility functions are not jointly identified in a general dynamic discrete choice model without additional restrictions. Here, $\rho = 0.9$ is fixed and γ is estimated subject to being constant over time. Our estimates and local rank checks are conditional on these restrictions and the remaining model assumptions; they do not establish identification of a specification with free ρ or unrestricted γ_t .

4 Demand estimates

Table 4 reports the two trouser SKUs under the historical price forecasts. Mean price coefficients are 2.58 and 2.22 for SKU 1560649 and 1560650, respectively. Estimated dispersions are 1.71 and 1.25, with conventional Wald p -values of 0.0076 and 0.0261.

Table 4: Demand estimates with historical price expectations

SKU	Trousers T14–16 1560649	Trousers T38–50 1560650
Purchase intercept α	−5.53 (0.45)	−4.91 (0.43)
Mean price coefficient $\bar{\beta}$	2.58 (1.17)	2.22 (0.90)
Price-coefficient dispersion σ_β	1.71 (0.64)	1.25 (0.56)
Early purchase shift δ_{12}	−0.32 (0.11)	−0.48 (0.10)
Waiting shift d (weeks 8–11)	−1.59 (0.06)	−1.56 (0.05)
Outside-option value γ	−4.48 (0.28)	−3.99 (0.22)
Customer–SKU histories	30,137	30,141
First purchases	2,450	2,662
Log likelihood	−13,714.69	−14,659.58
Conventional Wald p for σ_β	0.0076	0.0261
Implied share $\beta_i < 0$ (%)	6.57	3.81

Notes: Separate normal- β models for fourteen stores; $\rho = 0.9$. Waiting flow is $d\mathbf{1}\{8 \leq t < 12\}$. Price-transition beliefs are estimated using 2014 data and held fixed. Parentheses give unclustered observed-information standard errors, conditional on reconstructed prices and beliefs. Blue denotes numerical convergence and local-rank checks, not significance.

The negative δ_{12} lowers buying flow utility in weeks 1–2, conditional on price and type. The negative d lowers waiting flow utility in weeks 8–11. Waiting still includes the expected continuation value $\rho\mathcal{C}_t(z, p)$. The constant γ values absorbing exit, including buying elsewhere or forgoing purchase, relative to the normalized waiting-flow intercept.

The fitted normal distributions assign 6.57% and 3.81% of population mass to negative price coefficients. These types have buying flow utility that increases with price, a restriction of the normal specification that matters for the pricing exercise.

For each SKU, two EM starts agree at the interior solution. Scores, finite-difference derivatives, finer quadrature, and local rank checks validate the numerical results. Standard errors use the observed Hessian. Inference conditions on prices and historical beliefs, without clustering by store or accounting for their reconstruction and estimation uncertainty.

5 Dynamic seasonal pricing

The retailer chooses prices using the estimated demand and the same historical price beliefs. Customers observe today’s price and forecast future prices with the 2014 kernel; they are not given the retailer’s optimized future path. The firm anticipates their buying, waiting, and exit probabilities. This is a pricing exercise conditional on historical consumer beliefs, rather than an equilibrium in which those beliefs are required to reproduce the new policy.

5.1 The firm's state, expectations, and Bellman equation

Let $\mu_t(dz)$ be the expected number of customers of permanent type z still active at the start of week t , aggregated across stores. Initially, $\mu_1(dz) = N\phi(z) dz$, where $N = \sum_\ell N_\ell$. Given current relative price p , write $\pi_t^a(z, p; K)$ for the demand model's choice probability under the fixed historical kernels K . The firm's expectations integrate over both unobserved customer types and future choice shocks. For a deterministic price path, expected sales and the remaining type distribution satisfy

$$\mathbb{E}[Q_t \mid \mu_t, p_t, K] = \int \pi_t^B(z, p_t; K) \mu_t(dz), \quad \mu_{t+1}(dz) = \pi_t^W(z, p_t; K) \mu_t(dz). \quad (30)$$

Buying and exit remove customers from the active population. Waiting changes its composition: a scalar count of remaining customers is generally insufficient when price sensitivity is heterogeneous.

The retailer maximizes expected revenue net of VAT,

$$\max_{p_{1:T}} \mathbb{E} \left[\sum_{t=1}^T m(p_t) Q_t \mid \mu_1, K, \hat{\theta} \right], \quad m(p) = \frac{P^{\text{list}} p}{1 + \tau}, \quad \tau = 0.19. \quad (31)$$

Acquisition cost is sunk before the campaign; incremental unit cost and salvage value are zero, and availability is sufficient, as in the demand model. Each first purchase is treated as one unit.

Let a be the previous relative price, which bounds the next price from above. With $F_{T+1} = 0$, the expected-mass planning problem has Bellman equation

$$F_t(\mu, a) = \max_{p \in \mathcal{G}: p \leq a} \left\{ m(p) \int \pi_t^B(z, p; K) \mu(dz) + F_{t+1}(\pi_t^W(\cdot, p; K) \mu, p) \right\}. \quad (32)$$

The second term values customers who remain for future weeks. Thus today's price affects current revenue and the size and composition of future demand. This recursion optimizes deterministic markdown paths using expected population transitions. It does not claim to solve a policy that responds to random realized sales or unobserved stockouts.

5.2 Feasible policies and results

Prices are common across the fourteen stores and nonincreasing, with $\mathcal{G} = \{1, .9, .8, .7, .6, .5\}$ for both SKUs. These percentage-grid levels are available from the first week in the policy exercise; the special CLP 1,990 clearance price is outside this control grid. There are 6,188 feasible twelve-week paths per SKU. We solve (32) and independently enumerate all paths; both procedures must give the same revenue and optimal path. The benchmark uses the reconstructed, store-specific 2015 prices with identical demand parameters, beliefs, and customer counts.

Table 5: Optimal markdown paths with historical price beliefs

SKU	Optimal share of list price	Revenue (CLP million)			Gain (%)
		2015	List	Optimal	
1560649	100% (wk. 1–12)	11.34	12.21	12.21	7.7
1560650	100% (wk. 1–8); 90% (wk. 9–10); 80% (wk. 11–12)	14.06	14.34	14.34	2.0

Notes: Predictions for sampled customers, holding the historical price-belief kernels and outside option fixed. Revenue is net of 19% VAT. Acquisition cost is sunk, availability is sufficient, and one first purchase is treated as one unit. The 2015 benchmark evaluates store-specific reconstructed prices; the list benchmark keeps the list price throughout. Optimal paths are common across stores.

The optimal path keeps SKU 1560649 at its list price. SKU 1560650 uses a 10% markdown in weeks 9–10 and a 20% markdown in weeks 11–12; its improvement over keeping the list price is only CLP 2,444. Expected revenue increases 7.7% and 2.0%, respectively, relative to the reconstructed 2015 paths. A flat optimal path is permitted by the dynamic problem; markdowns are chosen only when their expected revenue contribution warrants them.

The fitted normal distributions for the trousers still assign 6.6% and 3.8% of initial customers to negative price coefficients. These types generate 39.8% and 20.6% of predicted purchases under the respective optimal paths, so their influence on pricing remains material.

The outside option and historical forecasting rule are held fixed under the new prices. The estimates do not identify how customers would learn a new retailer policy or how competitors would respond. The resulting revenue comparison is therefore conditional on these beliefs, the sampled customers, full availability, and the finite control grid.

6 Conclusion

The two-SKU analysis connects reconstructed posted prices, historical price forecasts, and latent purchase timing. Consumers observe the current price and integrate over future price states and choice shocks. EM retains the joint posterior over permanent types and unobserved exit weeks. The two trouser SKUs have positive estimated dispersion under conventional Wald tests, conditional on the specified price beliefs and timing controls.

The retailer’s dynamic problem balances current expected revenue against the value of customers who remain active. Under fixed historical beliefs, its optimal paths keep SKU 1560649 at list price and apply small late markdowns to SKU 1560650. Predicted net revenue rises 7.7% and 2.0%, respectively, relative to the model’s predictions under reconstructed 2015 prices. These are conditional policy calculations: historical payment proxies do not reveal customer beliefs, normal trouser heterogeneity retains a negative tail, and the exercise does not identify inventory constraints or belief adjustment to a new pricing rule.

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