Chapter 4

Nonlinear dynamical systems

4.1 Introduction

In this chapter we study nonlinear dynamical systems, which are the most frequent systems appearing in economic models and other applications, without limiting ourselves to the scalar case. Many of these models are well known in the economic literature and some of them will be analyzed in this chapter. Several nonlinear systems can't be analytically solved since there is no universal algebraic method providing a solution, whereas in the linear case, as it was seen in the previous chapter, it is possible. Nonetheless, this is not an obstacle to obtaining interesting conclusions about the model, as well as in the scalar case, the main analysis strategy is qualitative and is fundamentally founded in the information provided by the vector field. Even if the ideas are essentially the same, passing to higher dimensions makes more complex the problem, the theory becoming more sophisticated, but certainly much richer.

Another analysis strategy, which is complementary to the qualitative analysis, consists basically on linearize the system in order to apply the tools provided in the linear systems case. In this context, Hartman-Grobman Theorem, which is studied in the next section, has a central role.

This chapter consists of four sections. In the first one, some notions about vector fields and tangent vector, which are the basis of the qualitative analysis, are revisited. In Section 4.2 we present Hartman-Grobman Theorem, which is the fundamental tool to address nonlinear systems through linearization. Altogether, qualitative analysis, which is basically geometric, and the linearization, which is overall quantitative, are sufficient to plot the phase diagram of several nonlinear systems allowing us to obtain fundamental conclusions about the model. Then, to study systems with saddle point equilibriums, in Section 4.3, it is presented the Stable Manifold Theorem, analogous mathematical concept to the stable subspace in the linear systems theory. Finally, in Section 4.4, we analyze the existence of limit cycles and provide Poincaré-Bendixson Theorem. These results are very useful when it is time to study some cyclic behavior in economic models, as, for instance, Kaldor's nonlinear model for business cycle.

Let $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ be an open set and $F : \mathcal{U} \to \mathbb{R}^n$ a continuous function. We next define a nonlinear dynamical system.

Definition 4.1.1. A nonlinear first order dynamical system of dimension n is an equation of the form

$$\mathbf{x}^{'}(t) = F(t, \mathbf{x}(t)), \tag{4.1.1}$$

As before, we usually will dismiss the time variable as an argument of the state variable $\mathbf{x}$. If $F = (F_1, ..., F_n)$, System (4.1.1) can be written as the following set of n scalar differential equations:

$$\begin{array}{rcl} x_1^{'} & = & F_1(t, x_1, \ldots, x_n) \\ \vdots & \vdots & \vdots \\ x_n^{'} & = & F_n(t, x_1, \ldots, x_n), \end{array}$$

Under this setting, of course, at least one of the functions F_i, $i = 1, ..., n$ is nonlinear and all of them are continuous.

Observe that when $n = 1$, the system becomes scalar, and when F is linear and does not depend explicitly on t, becomes linear, i.e., $\mathbf{x}' = A\mathbf{x}$. Observe, as well, that by introducing the variable $x_{n+1} = t$ (and therefore $x'_{n+1} = 1$), the system becomes autonomous:

$$\mathbf{x}' = F(\mathbf{x}). \tag{4.1.2}$$

The following example shows the procedure above.

Example

Example 4.1.1. Consider the following nonautonomous system

$$\begin{aligned} x_1^{'} &= 2tx_1 + x_1x_2 \\ x_2^{'} &= x_1 + t^2x_2 + t^3. \end{aligned}$$

Let us introduce one more state variable by making $x_3 = t$. Plugging this into the equations above yields

$$x_1' = 2x_1x_3 + x_1x_2$$
$$x_2' = x_1 + x_2x_3^2 + x_3^3$$
$$x_3' = 1.$$

The system has gained one more state variable and therefore the dimension has increased. However, the procedure turns the system from non-autonomous to autonomous.

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VECTOR FIELD

As most economic models are formulated in terms of two variables (or can be reduced to two variables), we focus our attention on the bidimensional case of (4.1.2). As is usual in the literature, we write the system under the form

$$\begin{aligned} x_1' &= f(x_1, x_2) \\ x_2' &= g(x_1, x_2). \end{aligned} \tag{4.1.3}$$

Hence, the function F from (4.1.2) is given by $F = (f, g)$, where at least one of the functions, f or g, is nonlinear, and both are continuous. Let $I \subset \mathbb{R}$ be an open interval. We next define the solution of (4.1.3).

Definition 4.1.2. A solution of system (4.1.3) is a C^1 function $\mathbf{x} : I \to \mathbb{R}^2$, such that the system is satisfied point wise:

$$\forall\, t \in I \;:\; \mathbf{x}'(t) = (f(\mathbf{x}(t)), g(\mathbf{x}(t))).$$

Example

Example 4.1.2. The following system can be solved by applying what we learned before.

$$x_1^{'} = 2x_1$$
$$x_2^{'} = x_1^2 + 3x_2.$$

Functions $x_1(t) = e^{2t}$ and $x_2(t) = e^{3t} + e^{4t}$ are the corresponding solution of the equations. Indeed, $\forall\ t \in \mathbb{R}$ one has $x_1'(t) = 2e^{2t} = 2x_1(t)$ and $x_2'(t) = 3e^{3t} + 4e^{4t} = e^{4t} + 3e^{et} + 3e^{4t} = x_1^2(t) + 3x_2(t)$.

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The following figure shows the solution $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$ as a trajectory living in $\mathbb{R}^2$.

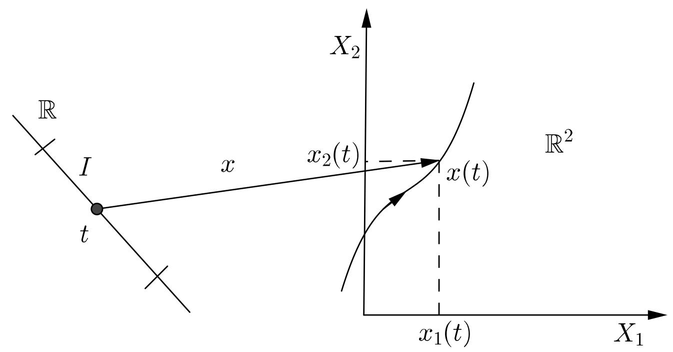


Figure 4.1 Trajectory of the solution in two dimensions

Since functions f and g are C^1, then for any initial condition $(t_0, \mathbf{x}(t_0))$ there exists an interval I over which there exists a unique solution in the sense of the Theorem 1 , studied in the first chapter. This theorem allows us to discard, for example, situations as the one illustrated in Figure 4.2.

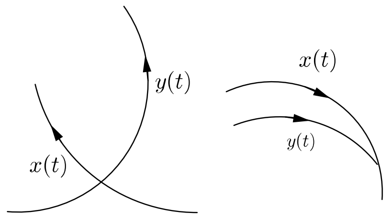


Figure 4.2 Trajectories can not intersect neither overlap

If there were an intersection or overlapping point, such a point could be taken as an initial condition, thus generating two different solutions for the same initial condition. This, of course, would violate Theorem 1.

The concept of vector field was already introduced in the firsts two chapters, but its usefulness is more evident for bidimensional systems. Before defining a vector field, let us explain first what a tangent vector is.

Definition 4.1.3. Let $\mathbf{x}(t) = (x_1(t), x_2(t))$ be a trajectory solution of (4.1.3). For every $t \in I$, the vector

$$\mathbf{x}'(t) = F(x_1(t), x_2(t)) = (x_1'(t), x_2'(t))$$

is a tangent vector to the trajectory.

Figure 4.3 shows a trajectory with several tangent vectors throughout it, for different times.

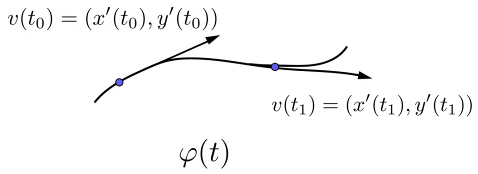


Figure 4.3 Tangent vectors at different times

A tangent vector $\mathbf{x}'(t)$ determines the direction in which the trajectory is moving at time t and the magnitude[1] $||\mathbf{x}'(t)||$ points out the rate of change with which the trajectory is being described at that moment.

Definition 4.1.4. Given system (4.1.3), the vector field $F = (f, g)$ is the infinite set of vectors

$$\mathbf{v}(P) = F(P) = (f(P), g(P)), \quad P \in \mathbb{R}^2.$$

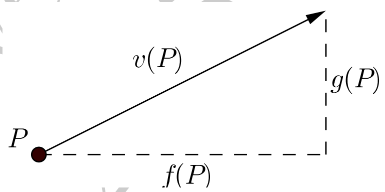
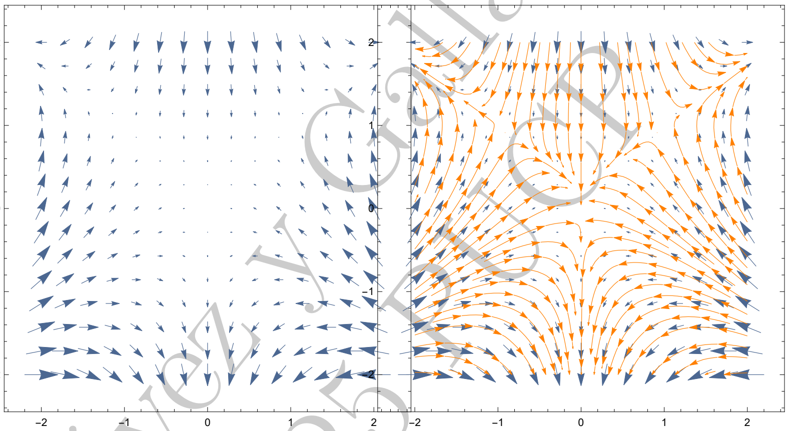


Figure 4.4 Vector field at P

For each point P, the vector field points in a certain fixed direction, determined by the pair of functions f and g, as it is shown in Figure 4.4.

[1] $||\mathbf{x}'(t)||$ is the Euclidean norm of vector $\mathbf{x}'(t)$, given by $\sqrt{(x_1'(t))^2 + (x_2(t)')^2}$.

Figure 4.5a shows that the vector field looks like an infinite set of vectors each one pointing out in different directions from a given point. On the other hand, Figure 4.5b shows that the vector field drives the trajectories of the system. In fact, for every point $P = \mathbf{x}(t)$ the tangent vector $\mathbf{x}'(t)$ belongs to the vector field and marks the path of the trajectory at that point. The set of all the trajectories is called the "phase diagram" and the space where these trajectories live is called the phase space.



(a) $\mathbf{v}(P) = (f(P), g(P))$ (b) Phase diagram

Figure 4.5 Vector field and trajectories

If $\mathbf{x}'(t) = (0, 0)$, the vector field does not point out in any direction and the trajectories does not move from the point $\mathbf{x}(t)$. On the other hand, if $x_1'(t) = 0$ but $x_2'(t) \neq 0$, the vector field is vertical. In this case, if $x_2'(t) > 0$, the trajectory moves upward, whereas if $x_2'(t) < 0$, it moves downward. Similarly, if $x_1'(t) \neq 0$ but $x_2'(t) = 0$, the vector field is horizontal. Thus, if $x_1'(t) > 0$, the trajectory moves to the right, and if $x_1'(t) < 0$, it moves to the left.

ISOCLINES AND ISOZONES

An isocline is a curve in $\mathbb{R}^2$ over which the vector field points out in the same direction. In particular, isoclines where the vector field has a vertical or horizontal direction, are very important for qualitative analysis. We denote these isoclines by $\mathcal{C}_v$ and $\mathcal{C}_h$, respectively. Formally

$$\mathcal{C}_v : \{P \in \mathbb{R}^2,\ f(P) = 0\}$$
$$\mathcal{C}_h : \{P \in \mathbb{R}^2,\ g(P) = 0\}.$$

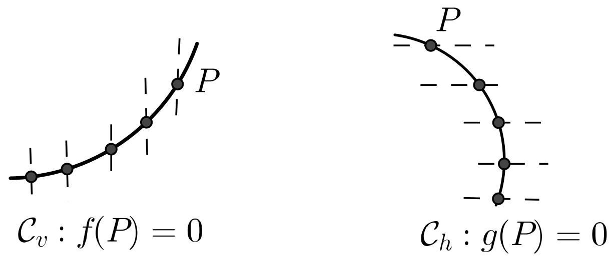


Figure 4.6 Isoclines for $f(P) = 0$ and $g(P) = 0$

Observe that these isoclines intersect each other at the stationary point, also known as the equilibrium. This is because each derivative is null at that point.

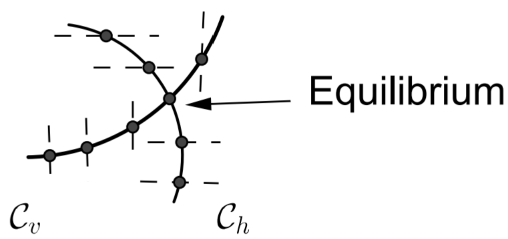


Figure 4.7 Equilibrium: intersection point of $\mathcal{C}_v$ and $\mathcal{C}_h$

The isocline $\mathcal{C}_v$ and $\mathcal{C}_h$ divide the phase space in two regions, as follows:

$$\begin{aligned}
\mathcal{R}_{v^-} &= \{P \in \mathbb{R}^2; \quad f(P) < 0\} \\
\mathcal{R}_{v^+} &= \{P \in \mathbb{R}^2; \quad f(P) > 0\} \\
\mathcal{R}_{h^-} &= \{P \in \mathbb{R}^2; \quad g(P) < 0\} \\
\mathcal{R}_{h^+} &= \{P \in \mathbb{R}^2; \quad g(P) > 0\}.
\end{aligned}$$

We will call each one of these regions as "isozones".

Example

Example 4.1.3. Consider the following system

$$\begin{aligned}
x_1' &= f(x_1, x_2) = x_1 + x_2^2 - 1 \\
x_2' &= g(x_1, x_2) = x_1 - x_2^2 + 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{C}_v: \ & f(x_1, x_2) = 0 \ \Leftrightarrow \ x_1 = -x_2^2 + 1 \\
\mathcal{C}_h: \ & g(x_1, x_2) = 0 \ \Leftrightarrow \ x_1 = x_2^2 - 1.
\end{aligned}$$

Thus, Isoclines $\mathcal{C}_v$ and $\mathcal{C}_h$ are parabolas with focal axis parallel to the OX axis (see Figure 4.8). Isozones are the following ones:

$$\begin{aligned}
\mathcal{R}_{v^-} &= \{(x_1, x_2) \in \mathbb{R}^2; \quad x_1 < -x_2^2 + 1\} \\
\mathcal{R}_{v^+} &= \{(x_1, x_2) \in \mathbb{R}^2; \quad x_1 > -x_2^2 + 1\} \\
\mathcal{R}_{h^-} &= \{(x_1, x_2) \in \mathbb{R}^2; \quad x_1 < x_2^2 - 1\} \\
\mathcal{R}_{h^+} &= \{(x_1, x_2) \in \mathbb{R}^2; \quad x_1 > x_2^2 - 1\}.
\end{aligned}$$

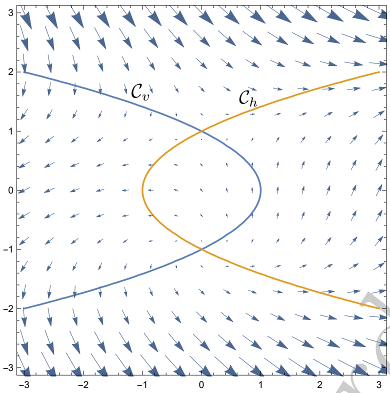


Figure 4.8 Isoclines and vector field for
$x_1' = x_1 + x_2^2 - 1$ and $x_2' = x_1 - x_2^2 + 1$

Observe that over each point of $\mathcal{C}_v$, the vector field is vertical and over each point of $\mathcal{C}_h$, it is horizontal. When the vector field crosses the isocline entering to the corresponding isozone, it maintains the orientation of the field.

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Figure 4.8 shows as well that there are two equilibria, the points where both isoclines intersect each other i.e., $P = (0, -1)$ and $Q = (0, 1)$. To plot the phase diagram with more acccuracy, we need to know how that trajectories behave in a neighborhood of these equilibriums. The fundamental tool to address this issue is Hartman-Grobman Theorem, which is developed in the next section.

PROBLEM SET

Exercise 4.1.1. Which of the following systems are nonlinear?

a) $x_1' = x_1, \quad x_2' = -2x_1^2 + x_2$

b) $x_1' = 3x_1 + 2x_2, \quad x_2' = x_2/t$

c) $x_1' = 2x_1 - x_1^2 - x_2, \quad x_2' = 2x_2 - x_1x_2.$

Exercise 4.1.2. The first two systems of Exercise 4.1.1 can be analytically solved. Find the solutions.

Exercise 4.1.3. Find the isoclines and plot the vector field for the nonlinear systems of Exercise 4.1.1.

Exercise 4.1.4. Consider the following figure.

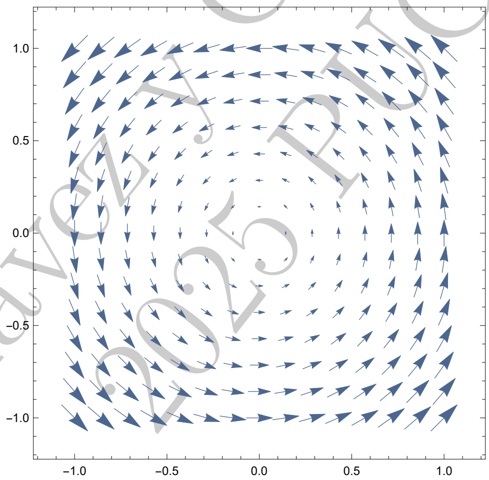

Figure 4.9 Vector field

Assume that the vector field of a certain function $F = (f, g)$ matches with the previous figure. How could be the trajectory of the system $x_1' = f(x_1, x_2)$, $x_2' = g(x_1, x_2)$? Give a pair of alternative solutions to $f(x_1, x_2)$ and $g(x_1, x_2)$.

4.2 Linearization and Hartman-Grobman Theorem

A constant solution of the system (4.1.3), $\mathbf{x}(t) = \mathbf{x}^* \; \forall \, t \in I$, is called an equilibrium solution, or stationary solution. An equilibrium solution, or, for short, an equilibrium, is such a state that if at some time the trajectory passes through $\mathbf{x}^*$, it won't move from there, unless an exterior perturbation upsets such an stationary state.

Following the definition, a constant solution $\mathbf{x}^* = (x_1^*, x_2^*)$ is an equilibrium if and only if it satisfies the following pair of algebraic equations:

$$0 = f(x_1, x_2)$$
$$0 = g(x_1, x_2).$$

Often the origin of the phase space is taken as the equilibrium $\mathbf{x}^*$. This is always possible due the change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$. By plugging this new variable into (4.1.3) we get

$$\mathbf{y}^{'} = F(\mathbf{y} + \mathbf{x}^*) \triangleq H(\mathbf{y}).$$

Hence, the point $\mathbf{x} = \mathbf{x}^*$ is an equilibrium of $\mathbf{x}' = F(\mathbf{x})$ if and only if $\mathbf{y} = \mathbf{0}$ is an equilibrium for $\mathbf{y}' = H(\mathbf{y})$.

Since the configuration of the phase diagram depends mostly on the nature of the equilibria, it is important to determine their stability i.e., if these are attractors, repellers, or saddle points, etc. This is done by analyzing the behavior of the neighbor trajectories. To accomplish this analysis, let us consider the following disturbance of the equilibrium $\mathbf{x}^*$: $\mathbf{x}(t) = \mathbf{x}^* + \boldsymbol{\eta}(t)$, where $\boldsymbol{\eta}(t)$ measures the difference between the equilibrium and a neighbor trajectory $\mathbf{x}$ at time t. This situation is illustrated in Figure 4.10.

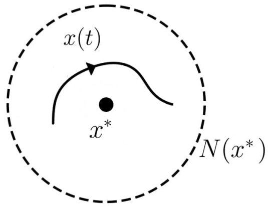

Figure 4.10 Perturbed equilibrium

Using a first order Taylor expansion at x^*

$$\begin{aligned}
\eta_1' = x_1' &\cong f(\mathbf{x}^*) + f_{x_1}(\mathbf{x}^*)(x_1 - x_1^*) + f_{x_2}(\mathbf{x}^*)(x_2 - x_2^*) \\
&= f_{x_1}(\mathbf{x}^*)(x_1 - x_1^*) + f_{x_2}(\mathbf{x}^*)(x_2 - x_2^*) \\
\eta_2' = x_2' &\cong g(\mathbf{x}^*) + g_{x_1}(\mathbf{x}^*)(x_1 - x_1^*) + g_{x_2}(\mathbf{x}^*)(x_2 - x_2^*) \\
&= g_{x_1}(\mathbf{x}^*)(x_1 - x_1^*) + g_{x_2}(\mathbf{x}^*)(x_2 - x_2^*).
\end{aligned}$$

In matrix form, this can be written as follows:

$$\boldsymbol{\eta}' \cong \begin{bmatrix} f_{x_1}(\mathbf{x}^*) & f_{x_2}(\mathbf{x}^*) \\ g_{x_1}(\mathbf{x}^*) & g_{x_2}(\mathbf{x}^*) \end{bmatrix} \boldsymbol{\eta} = J(\mathbf{x}^*)\boldsymbol{\eta},$$

where $J(\mathbf{x}^*)$ is the Jacobian matrix of F evaluated at $\mathbf{x}^*$, $J(\mathbf{x}^*) = J[F(x^*)]$, and $\boldsymbol{\eta} = [\eta_1, \eta_2]^T$. If the perturbation $\boldsymbol{\eta}$ is very small, by the continuity of the functions involved, it would be reasonable to expect that the dynamic for $\boldsymbol{\eta}$ should be given by

$$\boldsymbol{\eta}' = J(\mathbf{x}^*)\boldsymbol{\eta}.$$

So that, as $\mathbf{x}' = \boldsymbol{\eta}'$, when $\mathbf{x}$ is very close to $\mathbf{x}^*$, it is possible to analyze the non linear system $\mathbf{x}' = F(\mathbf{x})$ by analyzing the Associated Linear System (ALS) defined by

$$\mathbf{x}' = J(\mathbf{x}^*)\mathbf{x}. \tag{4.2.1}$$

Example

Example 4.2.1. Consider the system from Example 4.1.3. To obtain the equilibria, we need to solve the algebraic equations

$$\begin{aligned} x_1 + x_2^2 - 1 &= 0 \\ x_1 - x_2^2 + 1 &= 0. \end{aligned}$$

The solutions are given by $P^* = (0, -1)$ and $Q^* = (0, 1)$. For each equilibrium we set the corresponding ALS:

$$P^* \;\rightarrow\; ALS: \; \mathbf{x}' = J(P^*)\mathbf{x} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}\mathbf{x}$$

$$Q^* \;\rightarrow\; ALS: \; \mathbf{x}' = J(Q^*)\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}\mathbf{x}.$$

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When $J(\mathbf{x}^*)$ is nonsingular, the only equilibrium of (4.2.1) is the null equilibrium, $(0, 0)$. It is said that the equilibrium $\mathbf{x}^*$ from (4.1.2) and the equilibrium $(0, 0)$ from (4.2.1) are topologically equivalent if there exist neighborhoods of each equilibrium such that the trajectories lying on these neighborhoods have the same qualitative behavior. In that case, the stability of the equilibria $\mathbf{x}^*$

and $(0, 0)$ is the same. To formally define this ideas, we need to introduce the concept of homeomorphism.

Definition 4.2.1. Let E_1 and E_2 be two topological spaces. It is said that the map $h : E_1 \to E_2$ is an homeomorphism if it is continuous, bijective and its inverse $h^{-1} : E_2 \to E_1$ is continuous too. When there exists an homeomorphism between E_1 and E_2 we say that spaces E_1 and E_2 are homeomorphic.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are open sets of E_1 and E_2, respectively, and $h : \mathcal{U}_1 \to \mathcal{U}_2$ is an homeomorphism, h is called a local homeomorphism. It is said that $h : \mathcal{U}_1 \to \mathcal{U}_2$ preserves the orientation of the trajectories if a trajectory going from P to Q in $\mathcal{U}_1$ is transformed into a trajectory going from $h(P)$ to $h(Q)$ in $\mathcal{U}_2$.

Example

Example 4.2.2. Consider the linear systems $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{y}' = B\mathbf{y}$, where

$$A = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}.$$

Let $h(\mathbf{x}) = H\mathbf{x}$, where

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad H^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then, $B = H^{-1}AH$ and $e^{Bt} = H^{-1}e^{At}H$. Now, by setting $\mathbf{y} = h(\mathbf{x}) = H^{-1}\mathbf{x}$, it is obtained the following scheme:

$$\begin{array}{ccc} \mathbf{x}_0 & \to & \mathbf{x}(t) = e^{At}\mathbf{x}_0 \\ h\downarrow & & \downarrow\ h \\ \mathbf{y}_0 & \to & \mathbf{y}(t) = e^{Bt}\mathbf{y}_0. \end{array}$$

This outline shows that the map h transforms trajectories from the first system into trajectories of the second system, preserving their orientation. Moreover, since h is continuous with a continuous inverse, it is an homeomorphism. Thus, there is a correspondence between the trajectories of both systems.

$\diamond\diamond\diamond$

Definition 4.2.2. Let $\mathbf{x}^*$ be an equilibrium of (4.1.2). It is said that systems (4.1.2) and (4.2.1) are topologically equivalent in a neighborhood of $\mathbf{x}^*$ and $(0, 0)$ if there exists a local homeomorphism preserving the orientation of the trajectories between a neighborhood $\mathcal{U}_1$ of $\mathbf{x}^*$ and a neighborhood $\mathcal{U}_2$ of $(0, 0)$.

Essentially, this means that the trajectories of both systems have the same behavior in a neighborhood of their respective equilibria and thus the equilibria have the same stability (or instability).

Theorem 14 below, establishes under what conditions systems (4.1.2) and (4.2.1) are locally topologically equivalents around the equilibria $\mathbf{x}^*$ and $(0, 0)$, respectively. Before state it, it is necessary to define a preliminary concept. Given the system (4.1.2), it is said that an equilibrium $\mathbf{x}^*$ is hyperbolic if there is no eigenvalue of $J(\mathbf{x}^*)$ with non-zero real part.

Theorem 14. (Hartman-Grobman, H-G). If $\mathbf{x}^*$ is an hyperbolic equilibrium of (4.1.2), then systems (4.1.2) and (4.2.1) are topologically equivalents around $\mathbf{x}^*$ and $(0, 0)$, respectively.

A formal proof of Theorem 14 can be found, for instance, in Perko (2013) or Viana and Espinar (2021).

Example

Example 4.2.3. In regard with Example 4.2.1, we had

$$\text{Eigenvalues of } J(P^*): \ \frac{3}{2} + \frac{\sqrt{7}}{2}i, \ \frac{3}{2} - \frac{\sqrt{7}}{2}i$$
$$\text{Eigenvalues of } J(Q^*): \ -\frac{1}{2} + \frac{\sqrt{17}}{2}, -\frac{1}{2} - \frac{\sqrt{17}}{2}.$$

We realize that P^* and Q^* are hyperbolic equilibria. Now, by H-G Theorem, P^* is an unstable source and Q^* is an unstable saddle point. The phase diagram of the system is shown in Figure 4.11. In a very close neighborhood of P^* and Q^*, the trajectories behave as if the system were linear. This is precisely what Hartman Grobman Theorem predicts.

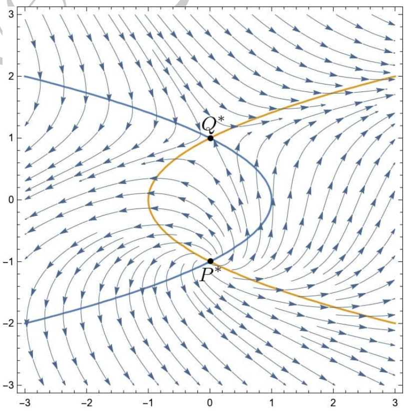

Figure 4.11 Phase diagram for $x_1' = x_1 + x_2^2 - 1$ and $x_2' = x_1 - x_2^2 + 1$.

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In the following we give some examples which combine the geometrical analysis, based on the vector field, and the qualitative analysis, based on Hartman-Grobman Theorem. In several cases, these tools allow us to plot with accuracy the phase diagram.

Examples

Example 4.2.4. Consider the nonlinear system

$$x_1' = -x_1$$
$$x_2' = 1 - x_1^2 - x_2^2.$$

The isoclines are

$$\mathcal{C}_v : x_1' = 0 \Rightarrow x_1 = 0$$
$$\mathcal{C}_h : x_2' = 0 \Rightarrow x_1^2 + x_2^2 = 1.$$

The isocline $\mathcal{C}_v$ is a vertical line passing through the coordinates origin, and the isocline $\mathcal{C}_h$ is a circumference of radius 1, centered in the coordinate origin. Both isoclines are shown in Figure 4.12

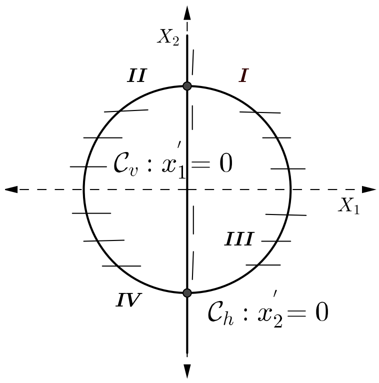

Figure 4.12 Isoclines for $x_1' = -x_1$ and $x_2' = 1 - x_1^2 - x_2^2$.

Isoclines $\mathcal{C}_v$ and $\mathcal{C}_h$ divide the phase space in four isozones. Denote the isozone in the right hand side of $\mathcal{C}_v$ by I and the one of the left hand side, by II. Likewise, denote the isozone outside $\mathcal{C}_h$ by III and the one inside by IV. More precisely,

$$\begin{array}{lcll} x_1^{'} < 0 & \equiv & x_1 > 0 & \text{Isozone I} \\ x_1^{'} > 0 & \equiv & x_1 < 0 & \text{Isozone II} \\ x_2^{'} < 0 & \equiv & x_1^2 + x_2^2 > 1 & \text{Isozone III} \\ x_2^{'} > 0 & \equiv & x_1^2 + x_2^2 < 1 & \text{Isozone IV.} \end{array}$$

Figure 4.13 shows the vector field in each one of these isozones.

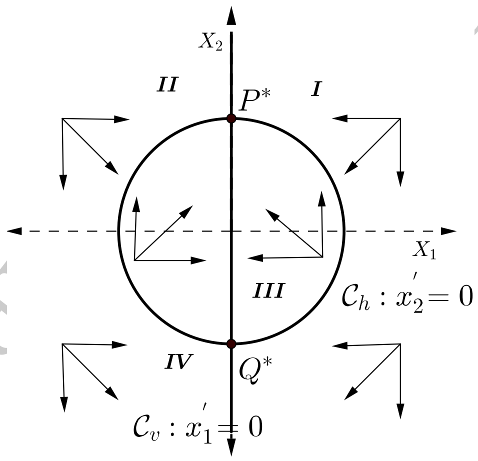

Figure 4.13 Vector field in each isozone.

The isoclines determine two equilibria, P^* and Q^* and the vector field suggests that P^* is an unstable saddle point and Q^* an stable attractor point. Let us verify this analytically.

From the equations $x_1^{'} = 0$ and $x_2{'} = 0$ we obtain $P^* = (0, -1)$

and $Q^* = (0, 1)$. Now the jacobian matrix is

$$J(x_1, x_2) = \begin{bmatrix} -1 & 0 \\ -2x_1 & -2x_2, \end{bmatrix}.$$

Then,

$$J(P^*) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \Rightarrow \lambda_1 = -1, \lambda_2 = 2, \text{ hyperbolic}$$
$$J(Q^*) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = -2, \text{ hyperbolic.}$$

The H-G theorem confirms our previous intuition about the nature of the equilibria. The phase diagram is shown in Figure 4.14.

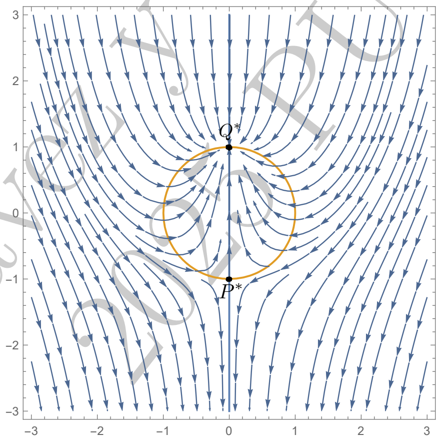

Figure 4.14 Phase diagram.

We observe that the trajectories coming from isozone I enter into the isozone IV horizontally. Once there, the trajectories go upward approaching the equilibrium Q^*, which is an attractor point. As these trajectories approach Q^*, they move away from the equilibrium P^*, which is a saddle point. On the other hand, the trajectories lying in the isozone I, and which do not go inside the isozone IV, either approach the attractor equilibrium, or move away the saddle equilibrium, remaining in isozone I. The phase diagram is symmetric with respect to $\mathcal{C}_v$ showing that the trajectories from isozone II have similar behavior to the trajectories in isozone I.

Figure 4.14 does not show the convergent trajectories to the saddle equilibrium P^*. By theory we know that if the system were linear, these trajectories would live in the stable subspace without leaving it. Nonetheless, when the system is nonlinear there is no such an stable subspace (neither an unstable subspace). As we will see later, the concept stable subspace (unstable subspace) is replaced with the one called "stable manifold" ("unstable manifold"), and it possesses essentially the same characteristics of the stable subspace (unstable subspace) in linear systems.

Let us give an example that shows complex eigenvalues.

Example 4.2.5. Let us consider the system

$$\begin{aligned} x_1' &= x_1^2 + x_2^2 - 2 \\ x_2' &= x_1^2 - x_2^2. \end{aligned}$$

The system has three isoclines:

$$\begin{aligned} \mathcal{C}_v &: x_1' = 0 \Rightarrow x_1^2 + x_2^2 = 2 \\ \mathcal{C}_{h1} &: x_2' = 0 \Rightarrow x_2 = -x_1 \\ \mathcal{C}_{h2} &: x_2' = 0 \Rightarrow x_2 = x_1. \end{aligned}$$

So there are two isoclines that are straight lines, $\mathcal{C}_{h1}$ and $\mathcal{C}_{h2}$, and one isocline that is a circumference, $\mathcal{C}_v$, as shown in Figure 4.15.

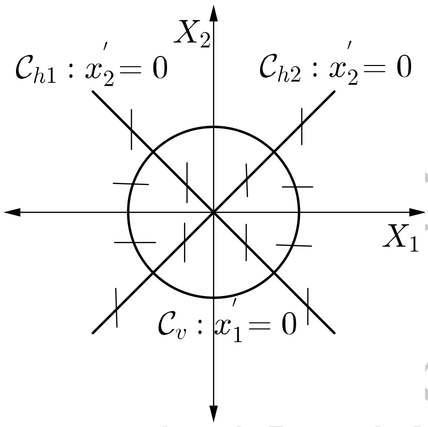

Figure 4.15 Isoclines

(a) Isocline $x_1^{'} = 0$:

- Isozone $x_1^{'} < 0 \equiv x_1^2 + x_2^2 < 2$. This isozone is located inside the circumference $\mathcal{C}_v$. In this region the vector field is horizontal and points out to the left.
- Isozone $x_1^{'} > 0 \equiv x_1^2 + x_2^2 > 2$. This isozone is located outside the circumference $\mathcal{C}_v$. In this region the vector field is horizontal and points out to the right.

(b) Isocline $x_2^{'} = 0$:

- Isozone $x_2^{'} < 0$. This isozone is determined by the union of the regions defined by the inequalities $x_2 > |x_1|$ and $x_2 < -|x_1|$. In this region the vector field is vertical and points out downward.

- Isozone $x_2' > 0$. This isozone is determined by the union of the regions defined by $-|x_1| < x_2 < |x_1|$. In this region the vector field is vertical and points out upward.

Figure 4.17 shows the vector field.

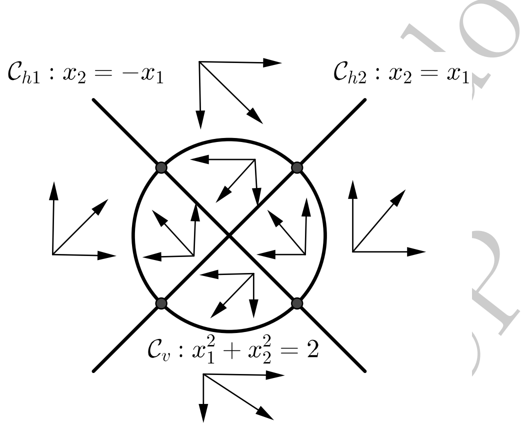

Figure 4.16 Vector field

As before we can give an insight about the trajectories from the vector field. We leave this for the reader. Let us apply the H-G theorem. By solving the equations

$$0 = x_1^2 + x_2^2 - 2$$
$$0 = x_1^2 - x_2^2,$$

we obtain four equilibria $P_1^* = (1, 1), P_2^* = (1, -1), P_3^* = (-1, 1)$ and $P_4^* = (-1, -1)$. The jacobian matrix is

$$J(x_1, x_2) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}.$$

Hence

$$J(P_1) = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}, \quad \lambda_1 = -2\sqrt{2}, \lambda_2 = 2\sqrt{2}, \text{ hyperbolic}$$

$$J(P_2) = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}, \quad \lambda_1 = 2 - 2i, \lambda_2 = 2 + 2i, \text{ hyperbolic}$$

$$J(P_3) = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}, \quad \lambda_1 = -2 - 2i, \lambda_2 = -2 + 2i, \text{ hyperbolic}$$

$$J(P_4) = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}, \quad \lambda_1 = -2\sqrt{2}, \lambda_2 = 2\sqrt{2}, \text{ hyperbolic.}$$

Since the equilibria are hyperbolic we are allowed to apply the H-G theorem. According to the theorem, P_1 and P_4 are unstable saddle equilibria; P_2 is unstable source equilibrium, and P_3 is a stable sink equilibrium. The phase diagram is shown in the Figure 4.17.

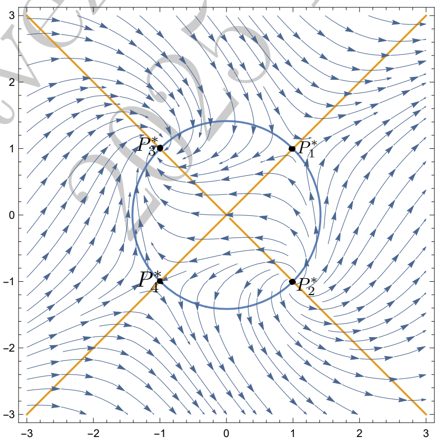

Figure 4.17 Vector field

◇◇◇

Example

Example 4.2.6. (Species competition). This well-known model provides a good opportunity to illustrate the concepts developed so far. The model describes the interaction between two species that compete for the same limited resource.

Let us take, for instance, rabbits and sheep, and assume that the ecosystem consists solely of these two species. Let us also assume that there is a single scarce resource essential for their subsistence. In order to survive, rabbits must compete with sheep for access to this resource. The population dynamics of these species evolve according to the logistic model.

Let x_1 and x_2 denote the populations of rabbits and sheep, respectively. The per capita growth rates of these populations are given by the following equations:

$$\frac{x_1'}{x_1} = 3 - x_1 - 2x_2, \tag{4.2.2}$$

$$\frac{x_2'}{x_2} = 2 - x_1 - x_2. \tag{4.2.3}$$

Equation (4.2.2) states that the per capita growth rate of rabbits decreases as both the rabbit population and the sheep population increase. In the absence of sheep, the rabbit population would evolve according to the standard logistic model. However, since sheep consume the same resource as rabbits, their presence negatively affects rabbit growth, which explains the negative coefficient of x_2. Similarly, the negative coefficient of x_1 reflects that as the number of rabbits increases, the limited resource must be shared among more individuals, thereby reducing the per capita growth rate.

An analogous interpretation applies for Equation (4.2.3), which describes the per capita growth dynamics of the sheep population. The constants 3 and 2 represent the intrinsic growth rates of rabbits and sheep, respectively, in the absence of competition.

By writing the equations as follows

$$x_1' = x_1(3 - x_1 - 2x_2)$$
$$x_2' = x_2(2 - x_1 - x_2),$$

we can identify the isoclines $\mathcal{C}_{v_1} : x_1 = 0, \mathcal{C}_{h_1} : x_2 = 0, \mathcal{C}_{v_2} : x_1 + 2x_2 = 3$ and $\mathcal{C}_{h_2} : x_1 + x_2 = 2$. Figure 4.18 shows thes isoclines and the vector field for the "meaningful isozones".[2]

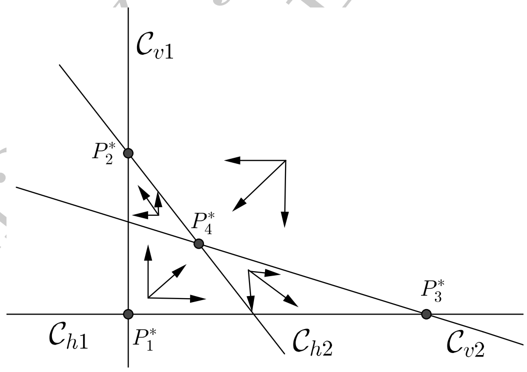

Figure 4.18 Vector field for species competition model

The equilibria are $P_1^* = (0,0), P_2^* = (0,2), P_3^* = (3,0)$ and

[2] Since x_1 and x_2 represent populations, a meaningful isozone only consider positive values for this variables.

$P_4^* = (1,1)$. The Jacobian matrix is

$$J(x_1, x_2) = \begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -x_2 & 2 - x_1 - 2x_2 \end{bmatrix}.$$

Hence

$$J(P_1^*) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \Rightarrow \lambda_1 = 3, \;\; \lambda_2 = 2, \;\text{hyperbolic}$$

$$J(P_2^*) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \Rightarrow \;\; \lambda_1 = -1, \;\; \lambda_2 = -2, \;\text{hyperbolic}$$

$$J(P_3^*) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix} \Rightarrow \;\; \lambda_1 = -3, \;\; \lambda_2 = -1, \;\text{hyperbolic}$$

$$J(P_4^*) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \Rightarrow \;\; \lambda_1 = -1 - \sqrt{2}, \lambda_2 = -1 + \sqrt{2}, \;\text{hyperbolic.}$$

By H-G Theorem, P_1^* is an unstable repulsive equilibrium, P_2^* and P_3^* are stable attractors and P_4^* is a saddle point equilibrium. The phase diagram is shown in Figure 4.19.

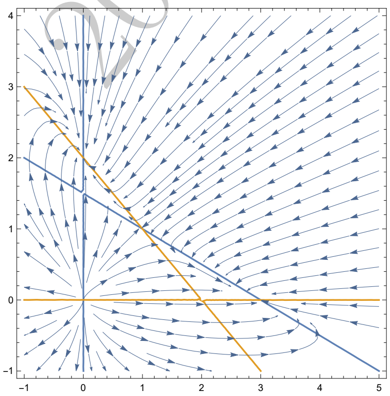

Figure 4.19 Species competition model

Conclusions of the model.

Based on the phase diagram it is possible to establish some interesting conclusions of the model. With warn that the horizontal axis corresponds to the rabbit population and the vertical axis corresponds to the sheep population. The meaningful region is the first quadrant, as both populations should be non-negative.

Observe that each trajectory which doesn't pass through the null equilibrium goes forward to one of the other three equilibria. In this context, it is appropriated to ask ourselves if each equilibrium is desirable scenario. Certainly the answer is no, as two of these three equilibria imply the extinction of at least one specie. The only equilibrium not leading to the extinction of one of both species is P_4^*, which is a saddle point. Thus, the initial condition for this model should be so that the trajectory will go to P_4^*. Contrary, species will become extinct in the long term.[3] The remaining question of how the initial condition should be placed in the phase space will be discussed in the next section. For the moment let us see what happens in the other scenarios.

If the initial condition is located at P_1^*, there will be neither rabbits nor sheep at the beginning, and, since P_1^* is an equilibrium, both species will never appear.[4] However, since P_1^* is a repeller, any small perturbation would be enough for the rabbits and sheep population starts growing. Nonetheless, in the long term, one of these populations will disappear, unless its trajectory goes forward

[3] Technically species never extinct, as the equilibrium is never reached. Why?

[4] As a matter of fact neither rabbits nor sheep can come out of nowhere.

to the saddle point equilibrium. Therefore, it is imperative to find a way to reach such equilibrium.

On the other hand, if the initial condition is a point far away from P_1^*, i.e., both rabbits and sheep populations are very big, these population will decrease over time. So a too big populations is not a sustainable situation in the long time. Presumably, since the resource is scarce, there is no just a competition between the species, but also between the members of each population itself.[5] Once again, it depends on the initial condition if some of both species will extinct or none of them (if the trajectory leads to the saddle point equilibrium).

Finally, if the initial condition would be located precisely at the saddle point equilibrium, or it is placed under a certain trajectory leading to P^*, then the population will remain coexisting in this scenario. Nevertheless, as the equilibrium is surrounded by trajectories moving away, a very small perturbation might drastically change the situation, and so the population will converge to a non desirable equilibrium.

$\diamond\diamond\diamond$

The following example, which could model several phenomena, is known in the literature as the Predator-Prey Model. This model shows a remarkable difference with the previous one, as the non-null equilibrium is not hyperbolic and, therefore, the H-G Theorem can not be applied to analyze its stability.[6] As we will see, the

[5] This is compatible with the logistic model idea.

[6] The other equilibrium is $(0, 0)$, which is not interesting for our purposes.

vector field suggests a circular behavior of the trajectories, leaving the doubt if the equilibrium is a center, a sink, or a source. In this context, some ad hoc arguments are going to be used to obtain meaningful conlcusions. It is surprising how the same behavior can be identified in a very famous economic model about salaries and employment, that we will analyze later. Even if the framework is different, the mathematical analysis is very similar.

Example

Example 4.2.7. (Predator-Prey model). The following model is known in the literature as the Predator-Prey model or the Lotka-Volterra model. It was introduced in 1925 by the American biophysicist Alfred Lotka, and, independently, the same year, by the Italian mathematician Vito Volterra. This model, unlike the previous ones, presents a non hyperbolic equilibrium, such that the H-G theorem can not be applied. The model describes the interaction between two population, one being the prey, and the other being the predator. For instance, we could think in salmons for the preys, and bears for the predators. Let us denote by x_1 the salmons population and by x_2 the bears population. The following are the hypothesis of the model.

H1. Salmon have unlimited resources, and therefore, without bears, its population (in accordance with Malthus) will grow exponentially at a constant rate $r_1 > 0$, that is, $x_1' = r_1 x_1$.

H2. Nonetheless, salmon's ecosystem is shared with bears, the predators. Therefore, the population of salmon decays in presence of bears (since salmon constitute a feeding source for bears). To be more specific, let us state that the growth rate x_1'/x_1 decreases linearly with respect to the population of bears x_2, $x_1'/x_1 = r_1 - ax_2$, where $a > 0$, quantifies the impact of bears on salmons. Thus, the dynamics of the population of salmon is

$$x_1' = r_1x_1 - ax_1x_2.$$

H3. The only feeding source for bears are salmons, so that, in the absence of salmon, bears population vanishes. Its population exponentially decays with constant rate $r_2 > 0$, that is, $x_2' = -r_2x_2$.

H4. The presence of salmon increases the population of bears. In this case we state that the growth rate x_2'/x_2 increases linearly with respect to the population of salmon x_1, $x_2'/x_2 = r_2 + bx_1$, where $b > 0$, quantifies the impact of salmon on bears. Thus, the dynamics of the population of bears is

$$x_2' = -r_2x_2 + bx_1x_2.$$

The model is therefore described by the following pair of differential equations:

$$x_1' = r_1x_1 - ax_1x_2 = x_1(r_1 - ax_2) \tag{4.2.4}$$

$$x_2' = -r_2x_2 + bx_1x_2 = x_2(-r_2 + bx_1). \tag{4.2.5}$$

The Isoclines are

$$\mathcal{C}_{v1} : x_1 = 0$$

$$\mathcal{C}_{v2} : x_2 = r_1/a$$

$$\mathcal{C}_{h1} : x_2 = 0$$

$$\mathcal{C}_{h2} : x_1 = r_2/b.$$

The vector field is shown in the Figure 4.20.

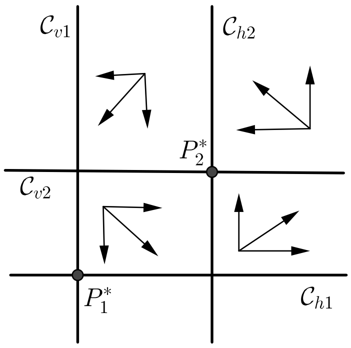

Figure 4.20 Vector field for Predator-Prey model.

The equilibria are $P_1^* = (0,0)$ and $P_2^* = (r_2/b, r_1/a)$, and the Jacobian matrix is

$$J(x_1,x_2) = \begin{bmatrix} r_1 - ax_2 & -ax_1 \\ bx_2 & -r_2 + bx_1 \end{bmatrix}.$$

Hence

$$J(P_1^*) = \begin{bmatrix} r_1 & 0 \\ 0 & -r_2 \end{bmatrix} \Rightarrow \lambda_1 = r_1, \ \ \lambda_2 = -r_2$$

$$J(P_2^*) = \begin{bmatrix} 0 & -\frac{a}{b}r_2 \\ \frac{b}{a}r_1 & 0 \end{bmatrix} \Rightarrow \ \lambda_1 = -\sqrt{r_1 r_2}i, \ \ \lambda_2 = \sqrt{r_1 r_2}i \ .$$

The equilibrium P^* is hyperbolic, and by H.G Theorem, it is a saddle point equilibrium. This equilibrium represents a non desirable scenario since it implies the extinction of both species. Since the equilibrium P_2^* is non hyperbolic, it is not possible to analyze its stability by using the H-G Theorem. Nonetheless, the vector field provides sufficient information to describe how the trajectories behave nearby this equilibrium. From the Figure 4.20, one can see that the trajectories seem to turn around P_2^*. They can be closed curves or spiral curves. To have some insight about the behavior of the trajectories, let us do the following algebra. From Equations (4.2.4) and (4.2.5) we have

$$\frac{dx_2}{dx_1} = \frac{x_2(-r_2 + bx_1)}{x_1(r_1 - ax_2)}.$$

By the method of separation of variables we have

$$\begin{aligned}
(r_1 - ax_2)\frac{dx_2}{x_2} &= (-r_2 + bx_1)\frac{dx_1}{x_1} \\
\left(\frac{r_1}{x_2} - a\right) dx_2 &= \left(-\frac{r_2}{x_1} + b\right) dx_1 \\
\int\left(\frac{r_1}{x_2} - a\right) dx_2 &= \int\left(-\frac{r_2}{x_1} + b\right) dx_1 \\
r_1 \ln x_2 - ax_2 &= -r_2 \ln x_1 + bx_1 + C \\
\ln x_2 - (a_1 x_2 + C_1) &= -r \ln x_1 + b_1 x_1, \qquad (4.2.6)
\end{aligned}$$

where $a_1 = a/r_1$, $C_1 = -C/r_1$, $r = r_2/r_1$ y $b_1 = b/r_1$. So we conclude that the solution trajectories to the model satisfy the algebraic equation (4.2.6). For x_1 fixed, a typical case of this equation is given in Figure 4.21.[7]

[7] The argument is the same even if the straight line doesn't cut the logarithmic curve.

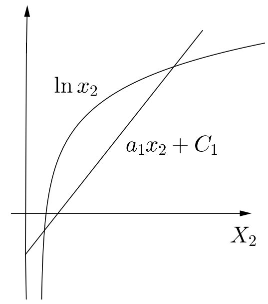

Figure 4.21 Implicit equation terms

If we keep constant x_1 in the right hand side of (4.2.6), then, if there exists x_2 satisfying the equation, we see from Figure 4.21 that this is possible at most for two different values of x_2. Hence, if we plot a vertical line in the phase space, this will only cut the graph of Equation (4.2.6) at most in two points. This fact discards the possibility of a spiral trajectory and so necessarily it must be a closed curve. A formal proof of this conclusion is given in [Hirsch and Smale (2012)].

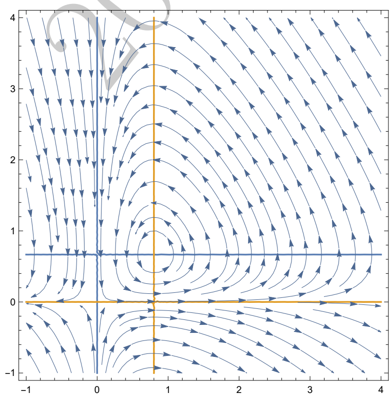

Figure 4.22 Predator-Prey phase diagram

Thus, we conclude that the origin is a saddle point and therefore unstable, while the non-zero equilibrium is a center.

Model conclusions.

From the phase diagram, we can describe the dynamics between the bear and salmon populations. When the salmon population begins to grow, the bear population also increases, as there are more salmon to eat. But when the bear population starts to grow, the salmon begin to disappear, since there are more bears eating them. As the salmon population starts to decline, the bear population also decreases because there are fewer salmon to eat. Finally, the decline in the bear population allows the salmon population to grow again, thus returning to the initial point, and the cycle repeats itself.

◊◊◊

In the following example, we present Goodwin's economic model which, as mentioned earlier, follows the same pattern as the Predator-Prey model. In fact, since the equations that determine the dynamics share the same mathematical structure, we should not expect to reach different conclusions from a qualitative point of view. The most logical question to ask at this point is: who is the prey and who is the predator in this model? Let us see.

Example

Example 4.2.8. (Goodwin's business cycle model). Richard Goodwin was an American economist (1913-1996) who dedicated most of his life to the study and understanding of the dynamics of capitalism. In his most known work "A Growth Cycle" published in 1967 in "Socialism, Capitalism and Economic Growth", Goodwin proposes a cyclic model, similar to Lotka-Volterra's with respect to the relation between capital and work.[8] The model aims to explain why capitalist economies exhibit endogenous cycles of expansion and contraction, without relying on external shocks. Goodwin focuses on the dynamic relationship between employment and income distribution.[9] The model suggests that economic cycles can be generated internally, driven by the distributional conflict between labor and capital. It is Marxist in spirit, highlighting class struggle as a core engine of economic dynamics.

Consider that the economy is made of capitalists and workers. To develop the model we introduce the following notation:

$$
\begin{aligned}
Y &: \quad \text{production} \\
K &: \quad \text{capital stock} \\
L &: \quad \text{labour force} \\
\ell &: \quad \text{labour demand} \\
a &: \quad \text{labour productivity} \\
w &: \quad \text{real wage rate.}
\end{aligned}
$$

The variable a measures the employee labor productivity, which is defined by

$$a(t) = \frac{Y(t)}{\ell(t)}. \tag{4.2.7}$$

Suppose that both labor productivity and labor force grow at

[8] Goodwin had a vision of the world inspired by Karl Marx's work.

[9] Observe that this situation is similar to the one about bears and salmons.

constant rates α and β, respectively:[10]

$$a(t) = a_0 e^{\alpha t} \tag{4.2.8}$$

$$L(t) = L_0 e^{\beta t}. \tag{4.2.9}$$

There is a constant capital-production relationship:

$$\frac{K(t)}{Y(t)} = \sigma. \tag{4.2.10}$$

Note that σ is the inverse of the marginal product of capital. The fundamental variables of Goodwin's model are the share of output allocated to workers, which we denote by u, and the share of the labor force that is actually employed, which we denote by v. Let us now examine these variables. The first variable is defined as follows:

$$u(t) = \frac{w(t)\ell(t)}{Y(t)}.$$

Due to (4.2.7), this can be written as

$$u(t) = \frac{w(t)}{a(t)}. \tag{4.2.11}$$

On the other hand, with respect to the employment rate, this one is defined by

$$v(t) = \frac{\ell(t)}{L(t)}. \tag{4.2.12}$$

From (4.2.11), by logarithmic differentiation, one has

$$\frac{u^{'}(t)}{u(t)} = \frac{w^{'}(t)}{w(t)} - \frac{a^{'}(t)}{a(t)}.$$

Due to (4.2.8), this can be written as

$$\frac{u^{'}(t)}{u(t)} = \frac{w^{'}(t)}{w(t)} - \alpha. \tag{4.2.13}$$

[10] For further simplification, we assume that all the population works.

Goodwin assumed that the wage rate has a positive linear relationship with the employment rate, that is,[11]

$$\frac{w'(t)}{w(t)} = -\gamma + \rho v, \quad \gamma, \rho > 0.$$

Taking into account (4.2.13), this can be written as

$$\frac{u'(t)}{u(t)} = -\gamma + \rho v - \alpha,$$

or equivalently

$$u'(t) = u(t)(-\gamma + \rho v(t) - \alpha). \tag{4.2.14}$$

Consider now the evolution of the employment rate v. From (4.2.12), once again by logarithmic differentiation, we have

$$\begin{aligned}\frac{v'(t)}{v(t)} &= \frac{\ell'(t)}{\ell(t)} - \frac{L'(t)}{L(t)} \\ &= \frac{\ell'(t)}{\ell(t)} - \beta. \end{aligned} \tag{4.2.15}$$

By isolating the variable ℓ in (4.2.7), we obtain

$$\ell(t) = \frac{Y(t)}{a(t)}.$$

From this, we have

$$\frac{\ell'(t)}{\ell(t)} = \frac{Y'(t)}{Y(t)} - \frac{a'(t)}{a(t)} \tag{4.2.16}$$

$$= \frac{Y'(t)}{Y(t)} - \alpha. \tag{4.2.17}$$

[11] Goodwin used a simplified version of the Phillips curve to express that the higher the employment rate, the higher the growth rate of real wages..

It is assumed that capitalists invest all their profits. Since u represents the share of output allocated to workers, denoting investment by I, we can write

$$K^{'}(t) = I(t) = (1 - u(t))Y(t). \tag{4.2.18}$$

Now, from (4.2.10) it follows that $K^{'}(t) = Y^{'}(t)\sigma$. Hence, by plugging this into (4.2.18), we obtain

$$\frac{Y^{'}(t)}{Y(t)} = \frac{1 - u(t)}{\sigma}. \tag{4.2.19}$$

By (4.2.16) this can be written as

$$\frac{\ell^{'}(t)}{\ell(t)} = \frac{1 - u(t)}{\sigma} - \alpha. \tag{4.2.20}$$

Finally, by combining equations (4.2.15) and (4.2.20), we obtain the dynamic equation for the variable v:

$$\frac{v^{'}(t)}{v(t)} = \frac{1 - u(t)}{\sigma} - \alpha - \beta,$$

or equivalently

$$v^{'}(t) = v(t)\left[\left(\frac{1}{\sigma} - (\alpha + \beta)\right) - \frac{1}{\sigma}u(t)\right]. \tag{4.2.21}$$

By combining equations (4.2.14) and (4.2.21), we derive the system of differential equations that defines Goodwin's model:

$$\begin{aligned} v^{'} &= f(v,u) = v\left[\left(\frac{1}{\sigma} - (\alpha + \beta)\right) - \frac{1}{\sigma}u\right] \\ u^{'} &= g(v,u) = u(-(\gamma + \alpha) + \rho v). \end{aligned}$$

Observe that these equations have the same structure than Lotka-Volterra's model. The employment rate, v, plays a role analogous to that of the prey in the Lotka–Volterra model, while the workers' share of output, u, corresponds to the predators. The equilibria of the model are $P_1 = (0, 0)$ and $P_2 = (v^*, u^*)$, where

$$v^* = \frac{\gamma + \alpha}{\rho}$$
$$u^* = 1 - (\alpha + \beta)\sigma.$$

The Jacobian matrix evaluated at the non-trivial equilibrium is:

$$J(P_2^*) = \begin{bmatrix} 0 & \frac{-(\alpha+\gamma)}{\sigma\rho} \\ \rho(1 - \sigma(\alpha + \beta)) & 0 \end{bmatrix}.$$

Hence, both the equations that define the model and the Jacobian matrix confirm that Goodwin's model exhibits the same structural behavior as the Lotka–Volterra model. The model concludes that, in a capitalist economy, wages and employment are two variables that are in constant conflict, yet mutually dependent. This interdependence gives rise to cyclical behavior in the dynamics of the economy.

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The following example, taken from [Strogatz (2018)], illustrates how a very small perturbation in the nonlinear term of a dynamical system can drastically alter the qualitative behavior of its trajectories. It also demonstrates that, once again, the geometric approach does not provide definitive conclusions about the stability of the equilibrium, and consequently, about the qualitative behavior of trajectories in its vicinity. Another valuable aspect of this example is that it introduces an analysis technique based on polar coordinates, which can be highly useful in similar contexts.

Example

Example 4.2.9. (Polar coordinates). Consider the following system:

$$\begin{aligned} x_1' &= x_2 + ax_1(1 - x_1^2 - x_2^2) \\ x_2' &= -x_1 + ax_2(1 - x_1^2 - x_2^2). \end{aligned}$$

There is a single equilibrium, $P^* = (0, 0)$. In fact, let us consider the algebraic system:

$$0 = x_2 + ax_1(x_1^2 + x_2^2) \tag{4.2.22}$$

$$0 = -x_1 + ax_2(x_1^2 + x_2^2), \tag{4.2.23}$$

with $a \neq 0$. Let $x_1 \neq 0$. Then, from (4.2.23), we deduce that $x_2 \neq 0$. Now, dividing both equations, we obtain

$$-\frac{x_1}{x_2} = \frac{x_2}{x_1} \quad \Rightarrow \quad -x_1^2 = x_2^2,$$

which is a contradiction. Clearly, the same contradiction arises if we start by assuming that $x_2 \neq 0$. Thus, the only equilibrium is the trivial equilibrium. The Jacobian matrix evaluated at the equilibrium point P^* is:

$$J(P^*) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since $\lambda_1 = -i$ and $\lambda_2 = i$, the equilibrium point P^* is non-hyperbolic and, therefore, the H–G Theorem cannot be applied. Observe that if the original system were linear, then, by the H–G Theorem, we would conclude that the null equilibrium is a center.

The form of the system suggests that we could use polar coordinates to analyze it. Let so $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Hence, $r^2 = x_1^2 + x_2^2$ and $\tan\theta = x_2/x_1$. Then

$$\begin{aligned} rr' &= x_1\left[x_2 + ax_1(x_1^2+x_2^2)\right] + x_2\left[-x_1 + ax_2(x_1^2+x_2^2)\right] \\ &= x_1x_2 + ax_1^2(x_1^2+x_2^2) - x_1x_2 + ax_2^2(x_1^2+x_2^2) \\ &= a(x_1^2+x_2^2)(x_1^2+x_2^2) \\ &= a(x_1^2+x_2^2)^2 \\ &= ar^4. \end{aligned}$$

From this, we obtain the equation:

$$r' = ar^3. \tag{4.2.24}$$

On the other hand, recall that, if $f(t) = \arctan w(t)$ then $f'(t) = w'(t)/(1+w^2(t))$. Thus,

$$\theta' = [\arctan(x_2/x_1)]' = \frac{x_1x_2' - x_2x_1'}{r^2}.$$

Making the appropriate substitution, we obtain:

$$\theta' = -1. \tag{4.2.25}$$

Therefore, the original system in cartesian coordinates x_1 and x_2 has been transformed into polar coordinates r and θ. The dynamic in this new coordinates system are given by the equations (4.2.24) and (4.2.25):

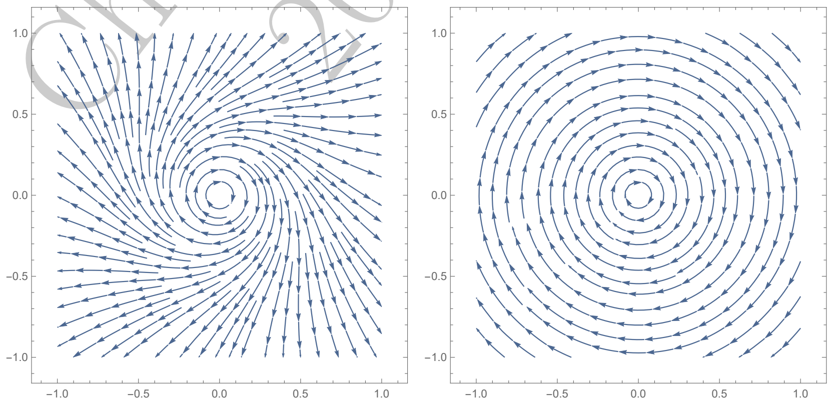
$$\begin{aligned} r' &= ar^3, \\ \theta' &= -1. \end{aligned}$$

Since the derivative of the angle of rotation is negative, the second equation indicates that the angle between the OX_1 axis and the trajectory is decreasing, that is, the motion is clockwise. Moreover, the angular velocity is constant and equal to 1.

On the other hand, since r is always positive, the first equation indicates that the derivative of r depends on the parameter a. Specifically, if $a > 0$, then r' is positive, causing r to increase. Consequently, the trajectories move away from the origin as time progresses. As they rotate clockwise around the origin, the trajectories spiral outward, and in this case, P^* behaves as a source. Conversely, if $a < 0$, the opposite occurs: the trajectories rotate clockwise, forming spirals that approach P^*. In this scenario, the equilibrium P^* acts as a sink.

Finally, if $a = 0$, then $r' = 0$, meaning r remains constant and the trajectories circle around P^* at a fixed distance. Here, P^* is a center.

These three cases are illustrated below.



(a) $a > 0$ (b) $a = 0$

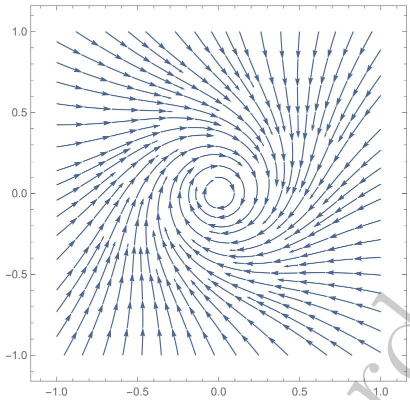


Figure 4.24 (c) $a < 0$

Observe that, if $a = 0$, the system is structurally unstable, as a very small perturbation in the value of this parameter changes its qualitative behavior. Whether the perturbation is positive or negative, the system's dynamics shift drastically.

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PROBLEM SET

Exercise 4.2.1. For each system, find the equilibria, the Jacobian matrix and the vector field:

a) $x_1' = 480x_1 - 8x_1^2 - 6x_1x_2, \quad x_2' = 2500x_2 - x_1^2x_2 - x_2^3.$

b) $x_1' = 4x_1 - 3x_1x_2, \quad x_2' = 3x_2 - x_1x_2.$

c) $x_1' = x_1x_2 - 2x_2, \quad x_2' = x_1x_2 - 2x_1.$

d) $x_1' = x_1 + x_2^2 - 1, \quad x_2' = x_1x_2 + x_1^2.$

Exercise 4.2.2. With respect to the systems in the previous exercise, apply the H-G Theorem to analyze the nature of the equilibria and plot the phase diagram.

Exercise 4.2.3. With respect to the system

$$x_1^{'} = ax_1 - x_1x_2,$$
$$x_2^{'} = bx_2 - x_1x_2,$$

where $a \neq 0$ and $b \neq 0$, solve the following items.

a) Prove that the system only possesses two equilibria, $P_1^* = (0,0)$ and $P_2^* = (b,a)$.

b) Determine the values of a and b for which P_1^* and P_2^* are repellers, attractors or saddle points.

Exercise 4.2.4. Repeat the analysis of the species competition model for the following systems:

a) $x_1^{'} = x_1(4 - x_1 - 2x_2), \quad x_2^{'} = x_2(6 - 2x_1 - x_2)$.

b) $x_1^{'} = x_1(6 - 2x_1 - x_2), \quad x_2^{'} = x_2(8 - 4x_1 - 2x_2)$.

Exercise 4.2.5. Consider the following variant of the Lotka–Volterra model, in which a logistic growth term has been introduced for the prey population:

$$x_1^{'} = ax_1\left(1 - \frac{x_2}{K}\right) - bx_1x_2,$$
$$x_2^{'} = cx_1x_2 - dx_2.$$

Complete the analysis of the model by examining the parameters c, d and K. Then, compare this model with the one presented in Example 4.2.7.

Exercise 4.2.6. Figure 4.25 shows the isoclines of a nonlinear system $x_1' = f(x_1, x_2)$, $x_2' = g(x_1, x_2)$

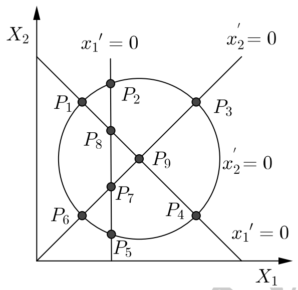

Figure 4.25 Isoclines for f? g?

a) Find the equilibria.

b) Find two candidate functions for f and g.

Exercise 4.2.7. Figure 4.26 shows the phase diagram of a system $x_1' = f(x_1, x_2)$, $x_2' = g(x_1, x_2)$. The straight line and the parabola are the isoclines and the parabola passes through $P = (4, -16/3)$.

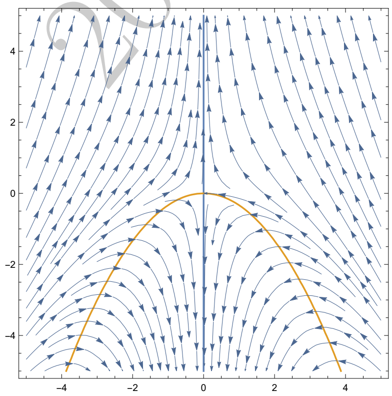

Figure 4.26 Phase diagram for f? and g?

Solve the following items:

a) Identify the equilibrium and analyze its nature based on the phase diagram.

b) Find f and g.

c) Apply the H-G Theorem to confirm your hypothesis in the first item.

4.3 Stable manifolds and stationary saddle solutions

In this section, we present the Stable Manifold Theorem. This theorem, one of the central results in the theory of dynamical systems, guarantees—under certain conditions—the existence of a stable subspace in the nonlinear case.

Although the most general version of the theorem and its proof are quite sophisticated, the result remains highly important and useful for qualitative analysis. Its statement is also somewhat expected, given that the H-G Theorem asserts that a nonlinear system behaves like its associated linear system in a neighborhood of a hyperbolic equilibrium.

Since many nonlinear models exhibit saddle point equilibria, the Stable Manifold Theorem becomes a powerful tool when selecting the trajectories that lead to the equilibrium in the long run, as discussed in the previous section.

Let $\mathbf{x}^*$ be an equilibrium of (4.1.2) and let E^s and E^u be the stable and unstable subspaces of the associated linear system $\mathbf{x}' = J(\boldsymbol{x}^*)\mathbf{x}$. Define $E^s(\mathbf{x}^*)$ and $E^u(\mathbf{x}^*)$ as follows:

$$E^s(\mathbf{x}^*) \triangleq E^s + \mathbf{x}^*,$$
$$E^u(\mathbf{x}^*) \triangleq E^u + \mathbf{x}^*.$$

Formally, $E^s(\mathbf{x}^*)$ and $E^u(\mathbf{x}^*)$ are known as the affine subspaces. Note that, if $\mathbf{x}^* = \mathbf{0}$, then $E^s(\mathbf{x}^*) = E^s$ and $E^u(\mathbf{x}^*) = E^u$.

Theorem 15. (Stable Manifold Theorem). Let $\mathbf{x}^*$ be a hyperbolic equilibrium of the system (4.1.2) and let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ containing $\mathbf{x}^*$. Assume that $F \in C^1(\mathcal{U})$ and let $\mathbf{x}(t; t_0, \mathbf{x}_0)$ denote the trajectory corresponding to the initial condition $\mathbf{x}_0 = \mathbf{x}(t_0)$. If $\mathbf{x}^*$ is a saddle point equilibrium, there exist differentiable curves denoted by $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$, known as manifolds [Guillemin and Pollack (1974)], defined in a neighborhood of $\mathbf{x}^*$, contained in $\mathcal{U}$, $\mathcal{N}(\mathbf{x}^*) \subset \mathcal{U}$. These manifolds $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$ are characterized in the following way:

1. $E^s(\mathbf{x}^*)$ and $E^u(\mathbf{x}^*)$ are tangent at $\mathbf{x}^*$ to $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$, respectively.

2. $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$ are given by

$$\begin{aligned} W^s_{loc}(\mathbf{x}^*) = \{\mathbf{x}_0 \in \mathcal{N}(\mathbf{x}^*) \ ; \ \mathbf{x}(t; t_0, \mathbf{x}_0) \in \mathcal{N}(\mathbf{x}^*) \ \forall\ t \geq t_0, \\ \lim_{t \to +\infty} \mathbf{x}(t; t_0, \mathbf{x}_0) = \mathbf{x}^*\}, \\ W^u_{loc}(\mathbf{x}^*) = \{\mathbf{x}_0 \in \mathcal{N}(\mathbf{x}^*) \ ; \ \mathbf{x}(t; t_0, \mathbf{x}_0) \in \mathcal{N}(\mathbf{x}^*) \ \forall\ t \leq t_0, \\ \lim_{t \to -\infty} \mathbf{x}(t; t_0, \mathbf{x}_0) = \mathbf{x}^*\}. \end{aligned}$$

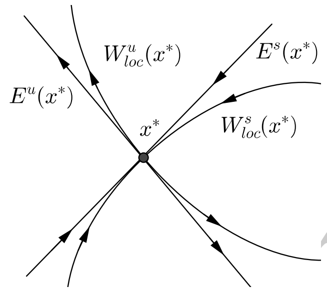

Figure 4.27 $W^s_{loc}(x^*)$ and $W^u_{loc}(x^*)$

The manifolds $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$ play the same role as the subspaces E^s and E^u in the linear case. As in that case, these sets are invariant; that is, for all t, we have $\mathbf{x}(t)(W^s_{loc}(\mathbf{x}^*)) \subset W^s_{loc}(\mathbf{x}^*)$ and $\mathbf{x}(t)(W^u_{loc}(\mathbf{x}^*)) \subset W^u_{loc}(\mathbf{x}^*)$. Moreover, trajectories starting in these sets either converge to the equilibrium $\mathbf{x}^*$ (in the stable case) or diverge from it (in the unstable case). In general, computing $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$ is highly nontrivial, except for certain specific systems such as the one presented below.

Examples

Example 4.3.1. The system

$$x_1^{'} = -4x_1,$$
$$x_2^{'} = 2x_2 - 3x_1^2$$

has a unique equilibrium at $\mathbf{x}^* = (0, 0)$. The associated linear system is:

$$(ALS): \quad \mathbf{x}' = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

Since the eigenvalues of the associated matrix are $\lambda_1 = -4$ and $\lambda_2 = 2$, the point $\mathbf{x}^*$ is a hyperbolic equilibrium. Thus, by the H-G Theorem, $\mathbf{x}^*$ is an unstable saddle point. The stable and unstable subspaces are:

$$E^s(\mathbf{x}^*) = \left\{ \mathbf{u} \in \mathbb{R}^2 : \ \mathbf{u} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \ \alpha \in \mathbb{R} \right\} \quad \rightarrow \quad \text{axis } X_1,$$

$$E^u(\mathbf{x}^*) = \left\{ \mathbf{v} \in \mathbb{R}^2 : \ \mathbf{v} = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \ \alpha \in \mathbb{R} \right\} \quad \rightarrow \quad \text{axis } X_2.$$

Given the initial condition $\mathbf{x}_0 = (x_{10}, x_{20})$, the solution to the system is:

$$x_1(t) = x_{10} e^{-4t},$$

$$x_2(t) = \left(x_{20} - \frac{3}{10} x_{10}^2 \right) e^{2t} + \frac{3}{10} x_{10}^2 e^{-8t}.$$

From this expression, it is clear that:

$$W^s_{loc}(\mathbf{x}^*) = \left\{ (x_1, x_2) \in \mathcal{N}(0,0); \ x_2 = \frac{3}{10} x_1^2 \right\},$$

$$W^u_{loc}(\mathbf{x}^*) = \{ (x_1, x_2) \in \mathcal{N}(0,0); \ x_1 = 0 \} = E^u(\mathbf{x}^*).$$

Indeed, in order for the solution $\mathbf{x}(t) = (x_1(t), x_2(t))$ to converges to $\mathbf{0}$ as $t \to \infty$, it is necessary to cancel the term $\left(x_{20} - \frac{3}{10} x_{10}^2 \right)$.

On the other hand, choosing the initial condition (x_{10}, x_{20}) on the unstable manifold should ensure that $\lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{0}$. However, since the term $\frac{3}{10} x_{10}^2 e^{-8t}$ diverges as $t \to -\infty$, we must have $x_{10} = 0$.

From this observation, and recalling that the stable and unstable subspaces are tangent to the stable and unstable manifolds, respectively, we can obtain the following figure.

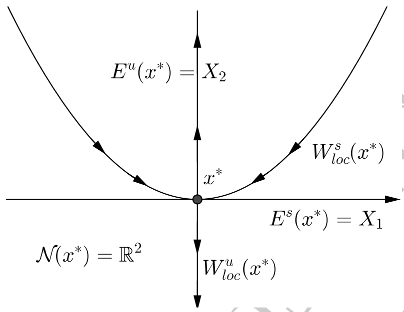

Figure 4.28 $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$

Therefore, the trajectories will diverge following axis X_2 and will converge to $(0,0)$ if they belong to the parabola $x_2 = (3/10)x_1^2$.

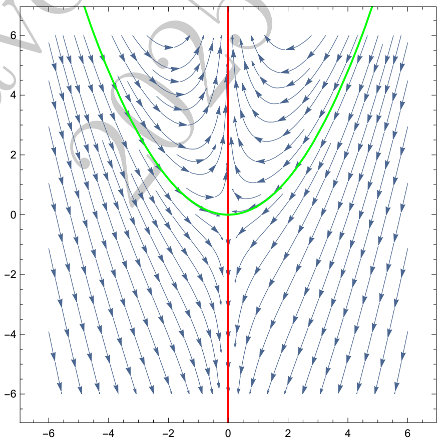

Figure 4.29 Phase diagram including the stable and unstable manifolds

Fortunately, in the previous example we were able to obtain analytical solutions, which allowed us to explicitly determine the stable and unstable manifolds. However, in most cases, it is not possible to find $W^s_{loc}(\mathbf{x}^*)$ and $W^u_{loc}(\mathbf{x}^*)$ analytically. The following example illustrates this through a nonlinear dynamical system in which the stable manifold can only be computed using numerical methods. Let us take a look.

Example 4.3.2. A prestigious publishing company has introduced a dynamic new strategy to boost sales by distributing coupons. Let x_1 represent the number of coupons in circulation and x_2 the quantity of books demanded. The company establishes a threshold ratio $r = \omega/\sigma$, such that when a consumer purchases more than r books at price σ, they receive coupons. Below this threshold, no new coupons are distributed, and the existing coupon stock decays at a rate ω. This behavior is captured by the following equation:

$$x_1' = \sigma x_2 - \omega.$$

With respect to demand behavior, it is assumed that the number of books purchased increases with the number of available coupons, and decreases, at the rate $\delta > 0$, with the quantity of books already sold. Accordingly, the behavior of x_2 is given by:

$$x_2' = x_1^2 - \delta x_2.$$

Combining both equation above, the nonlinear system becomes:

$$\begin{aligned} x_1' &= \sigma x_2 - \omega, \\ x_2' &= x_1^2 - \delta x_2. \end{aligned}$$

The equilibria of this system are $P_1^* = \left(-\sqrt{\delta\omega/\sigma}, \omega/\sigma\right)$ and $P_2^* = \left(\sqrt{\delta\omega/\sigma}, \omega/\sigma\right)$. We focus on P_2^*, since $x_1 < 0$ has no economic interpretation. The Jacobian of the system at P_2^* is:

$$J(P_2^*) = \begin{bmatrix} 0 & \sigma \\ 2\sqrt{\delta\omega/\sigma} & -\delta \end{bmatrix}.$$

Since $\det(J(P_2^*)) = -2\sigma\sqrt{\delta\omega/\sigma} < 0$, the equilibrium P_2^* is a saddle point. Thus, by the Stable Manifold Theorem, there exists a local stable manifold $W^s_{loc}(P_2^*)$ such that trajectories starting on it converge to the equilibrium.

Figure 4.30 shows the phase diagram, while Figure 4.31 highlights the stable manifold.

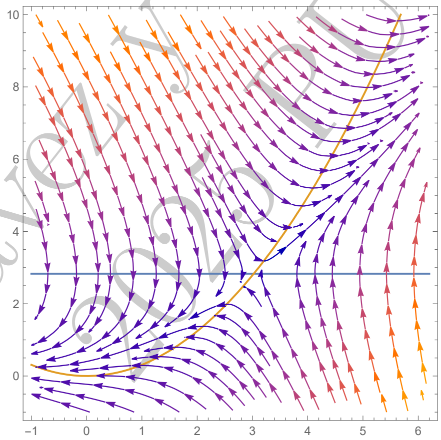

Figure 4.30 Phase diagram for the coupon-based book market

The equilibrium point P_2^* lies at the intersection of the isoclines $x_2 = x_1^2/\delta$ and $x_2 = \omega/\sigma$. One might intuitively expect that lowering ω, which defines the coupon threshold, would lead to increase in book sales. However, the equilibrium $x_2^* = \omega/\sigma$ actually increaes with ω.

This apparent paradox is clarified by studying the global dynamics. A higher ω lifts the equilibrium but reduces the basin of attraction, leading many trajectories toward extinction ($\lim_{t\to\infty} x_1(t) = x_2(t) = 0$). Lowering ω may result in more explosive dynamics, with feedback loops between coupon issuance and demand pushing both variables upward. Thus, although equilibrium values are higher with greater ω, the long-run outcomes may be worse for most initial conditions.

From a policy perspective, the company might want to avoid scenarios with unbounded book demand. In that case, the stable manifold $W^s_{loc}(P_2^*)$ becomes essential: if the initial condition lies on it, the system converges smoothly to the desired equilibrium.

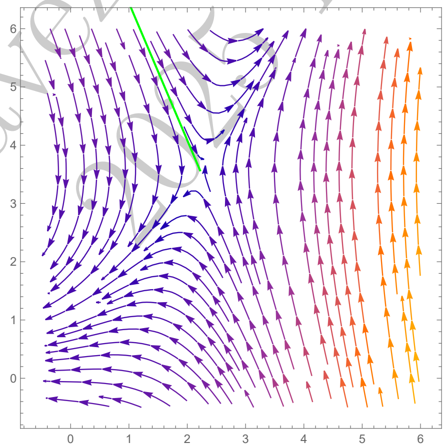

Figure 4.31 Stable manifold $W^s_{loc}(P_2^*)$

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PROBLEM SET

Exercise 4.3.1. Explain the advantage of finding the stable manifold from an economic point of view. Give an example of a model where the equilibrium(a) is (are) not desirable. Where should be the initial conditions with respect to W^s_{loc} and W^u_{loc} when the equilibrium is desirable? Give an example of such a case.

Exercise 4.3.2. Solve the following dynamic systems and provide the stable and unstable manifolds.

a)

$$x_1' = x_1 - x_2^3,$$
$$x_2' = -2x_2.$$

b)

$$x_1' = x_1 - x_2^2,$$
$$x_2' = -x_2.$$

Exercise 4.3.3. Recall the nonlinear dynamical system obtained in Example 4.3.2. Consider the following modification:

$$x_1' = \sigma x_2 - \omega,$$
$$x_2' = \alpha x_1 - \delta x_2^2,$$

with σ and ω positives. Provide the intuition of this model (meaning of each parameter) and the difference with the original model. Finally, explain why the Stable Manifold Theorem still can be applied for this model.

4.4 Limit cycles and periodic solutions

The Hartman–Grobman Theorem is a powerful tool for analyzing hyperbolic equilibria of nonlinear dynamical systems. When equilibrium points are nonhyperbolic, however—as in the classical predator–prey model—the linear approximation becomes inconclusive, and the analysis must rely on more ad hoc qualitative methods. In such cases, it is often observed that trajectories exhibit cyclical behavior, with solutions tracing closed orbits around a nontrivial equilibrium, typically identified as a center.

To study these phenomena, we begin by introducing several foundational definitions that will serve as the basis for the subsequent analysis. We then present a negative criterion: a set of conditions under which the existence of closed orbits can be ruled out. This is followed by the introduction of limit cycles, a special class of isolated closed trajectories that play a central role in nonlinear dynamics. Building on this, we state the Poincaré–Bendixson Theorem, which provides a positive criterion: under appropriate conditions, it guarantees the existence of closed orbits in two-dimensional systems. This result is of fundamental importance in the qualitative theory of differential equations and has broad applications, including in economic models that display cyclical behavior, which are studied at the end of this section.

Definition 4.4.1. Consider the autonomous system

$$\mathbf{x}' = F(\mathbf{x}),$$

and let $\mathbf{x}(t)$ be a solution with initial condition $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$.

The *orbit* through $\mathbf{x}_0$ is the image of this solution, defined by

$$\mathcal{O}(\mathbf{x}_0) = \{\mathbf{x}(t) \in \mathbb{R}^n : t \in \mathbb{R}\}.$$

The *positive orbit* and *negative orbit* through $\mathbf{x}_0$ are, respectively,

$$\mathcal{O}^+(\mathbf{x}_0) = \{\mathbf{x}(t) \in \mathbb{R}^n : t \geq 0\},$$
$$\mathcal{O}^-(\mathbf{x}_0) = \{\mathbf{x}(t) \in \mathbb{R}^n : t \leq 0\}.$$

Unless otherwise specified, we shall simply use the term *orbit* to refer to the positive orbit $\mathcal{O}^+(\mathbf{x}_0)$.

Definition 4.4.2. Consider the system

$$x_1' = f(x_1, x_2),$$
$$x_2' = g(x_1, x_2),$$

where f and g are continuous functions with continuous first-order partial derivatives.

A solution $\mathbf{x}(t) = (x_1(t), x_2(t))$ is called *periodic* if there exists a constant $T > 0$ such that

$$x_1(t+T) = x_1(t) \quad \text{and} \quad x_2(t+T) = x_2(t), \qquad \forall\, t \in \mathbb{R}.$$

The number T is called a *period* of the solution. If T is a period, then so is nT for every $n \in \mathbb{N}$. The smallest positive period, when it exists, is referred to as the *minimal period*.

Geometrically, a periodic trajectory corresponds to a closed orbit in the phase plane.

Definition 4.4.3. Consider the autonomous system

$$\mathbf{x}' = F(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n. \tag{4.4.1}$$

We say that (4.4.1) is a *gradient system* if there exists a continuously differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbf{x}' = -\nabla\varphi(\mathbf{x}), \tag{4.4.2}$$

with $\nabla\varphi \not\equiv 0$.

Remark 4. According to Definition 4.4.3, the condition $\nabla\varphi \not\equiv 0$ rules out the trivial case in which every trajectory is stationary.

Restricting to the two-dimensional case, the system (4.4.1) takes the form

$$x_1' = f(x_1, x_2),$$
$$x_2' = g(x_1, x_2).$$

Consequently, the gradient structure (4.4.2) is equivalent to

$$f = -\frac{\partial\varphi}{\partial x_1}, \qquad g = -\frac{\partial\varphi}{\partial x_2}.$$

Since $\varphi \in C^1(\Omega \subset \mathbb{R}^2)$, Schwarz's theorem (see Tao (2022)) yields

$$\frac{\partial}{\partial x_2}\left(\frac{\partial\varphi}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial\varphi}{\partial x_2}\right). \tag{4.4.3}$$

Therefore,

$$\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_1}.$$

This condition completely characterizes gradient systems in the plane. A fundamental property of such systems is the impossibility of closed orbits, i.e., nontrivial periodic trajectories.

Theorem 16. Gradient systems do not admit closed orbits; in particular, they cannot have periodic solutions.

Proof. Suppose, for contradiction, that a gradient system in $\mathbb{R}^n$ admits a closed orbit Γ. That is, $\mathbf{x}' = -\nabla\varphi(\mathbf{x})$ for some C^1 function φ, and there exists $T > 0$ such that $\mathbf{x}(T) = \mathbf{x}(0)$.

Along the trajectory, we compute the change in φ over one period:

$$\Delta\varphi = \varphi(\mathbf{x}(T)) - \varphi(\mathbf{x}(0)) = \int_0^T \frac{d\varphi}{dt}\,dt. \tag{4.4.4}$$

Since $\mathbf{x}(T) = \mathbf{x}(0)$, we must have $\Delta\varphi = 0$.

On the other hand, by the chain rule,

$$\frac{d\varphi}{dt} = \nabla\varphi \cdot \mathbf{x}'.$$

Substituting $\mathbf{x}' = -\nabla\varphi$, we obtain

$$\begin{aligned}\Delta\varphi &= \int_0^T \nabla\varphi \cdot \mathbf{x}'\,dt\\ &= -\int_0^T \mathbf{x}' \cdot \mathbf{x}'\,dt\\ &= -\int_0^T \|\mathbf{x}'(t)\|^2\,dt.\end{aligned}$$

Since $\nabla\varphi \not\equiv 0$ (Remark 4), the vector field $\mathbf{x}'$ is not identically zero, and therefore

$$\Delta\varphi = -\int_0^T \|\mathbf{x}'(t)\|^2\,dt < 0. \tag{4.4.5}$$

Equations (4.4.4) and (4.4.5) contradict each other. Thus, no closed orbit can exist in a gradient system, and consequently such systems admit no periodic trajectories. □

Example

Example 4.4.1. Consider the dynamical system

$$\mathbf{x}' = (x_1', x_2') = (-2x_1, -2x_2)\,. \tag{4.4.6}$$

We seek a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbf{x}' = -\nabla\varphi$. From the first component,

$$-\frac{\partial \varphi}{\partial x_1} = -2x_1 \quad \Longrightarrow \quad \frac{\partial \varphi}{\partial x_1} = 2x_1.$$

Integrating with respect to x_1,

$$\varphi(x_1, x_2) = x_1^2 + g(x_2), \tag{4.4.7}$$

for some function g depending only on x_2. Similarly, from the second component,

$$-\frac{\partial \varphi}{\partial x_2} = -2x_2 \quad \Longrightarrow \quad \frac{\partial \varphi}{\partial x_2} = 2x_2,$$

which upon integration yields

$$\varphi(x_1, x_2) = x_2^2 + h(x_1), \tag{4.4.8}$$

for some function h depending only on x_1. Comparing both expressions (4.4.7) and (4.4.8), we conclude that

$$\varphi(x_1, x_2) = x_1^2 + x_2^2 + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Thus, (4.4.6) is indeed a gradient system with potential function $\varphi(x_1, x_2) = x_1^2 + x_2^2 + C$. According to Theorem 16, such systems do not admit closed orbits, so (4.4.6) has no periodic trajectories.

Another very useful negative criterion, based on Gauss-Ostrogradski Theorem [Zorich (2016)], is Bendixson negative criterion, stated below.

Theorem 17. (Bendixson's negative criterion). Let the planar system

$$\mathbf{x}' = F(\mathbf{x}) = (f(x_1, x_2), g(x_1, x_2)) \tag{4.4.9}$$

be defined on a connected region $\Omega \subset \mathbb{R}^2$, where $F \in C^1(\Omega)$. If

$$\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2} > (<)\ 0 \quad \text{for all } \mathbf{x} \in \Omega,$$

then the system admits no closed orbits contained in Ω.

Proof. Suppose, by contradiction, that there exists a closed orbit Γ entirely contained within the region $\Omega \subset \mathbb{R}^2$, and let $\mathcal{D} \subset \Omega$ denote the region enclosed by Γ.

Since $F = (f, g)$ is continuously differentiable on Ω, the divergence theorem (Gauss–Ostrogradsky) applies and yields:

$$\oint_\Gamma F \cdot \mathbf{n}\, dr = \iint_{\mathcal{D}} \left(\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2} \right) dx_1\, dx_2, \tag{4.4.10}$$

where:

- Γ is a positively oriented closed curve,[12]
- $\mathbf{n}$ is the outward unit normal vector to Γ,[13]

[12] A curve Γ is said to be "positively oriented" in $\mathbb{R}^2$ if it is traversed in the counterclockwise direction. Equivalently, as one moves along the curve, the region enclosed by it lies to the left. This ensures that the outward-pointing normal vector $\mathbf{n}$ points away from the enclosed region.

[13] If Γ is parametrized as $\mathbf{z}(t) = (z_1(t), z_2(t))$, then $\mathbf{n}(t) = \frac{1}{\|\mathbf{z}'(t)\|}(-z_2'(t), z_1'(t))$.

- dr is the arc length differential,[14]
- $dx_1\, dx_2$ is the standard differential area element in $\mathbb{R}^2$.

Now, since Γ is a closed orbit of the autonomous system

$$\mathbf{x}' = F(\mathbf{x}),$$

the vector field $F(\mathbf{x})$ is tangent to the trajectory at every point on Γ. Consequently, since $\mathbf{n}(t)$ is orthogonal to the tangent vector at each point, we have

$$F(\mathbf{z}(t)) \cdot \mathbf{n}(t) = 0 \quad \text{for all } t,$$

and thus the line integral vanishes:

$$\oint_\Gamma F \cdot \mathbf{n}\, dr = \int_a^b F(\mathbf{z}(t)) \cdot \mathbf{n}(t)\, \|\mathbf{z}'(t)\|\, dt = 0.$$

However, if the divergence of F satisfies

$$\nabla \cdot F = \frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2} > 0 \quad (\text{or } < 0) \quad \text{throughout } \Omega,$$

then the right-hand side of (4.4.10) must be strictly positive (or strictly negative), since the integrand does not change sign and the region $\mathcal{D}$ has nonzero area.

This leads to a contradiction. Therefore, no closed orbit can exist in Ω. □

One last very famous and useful result, which purpose is to discard the existence of closed orbits is the following theorem.

[14] The arc length differential is given by $dr = \|\mathbf{z}'(t)\|\, dt$, where $\mathbf{z}(t)$ parametrizes the curve Γ.

Theorem 18. (Bendixson–Dulac negative criterion). Let

$$
\begin{aligned}
x_1' &= f(x_1, x_2),\\
x_2' &= g(x_1, x_2)
\end{aligned}
$$

be a planar dynamical system with $f, g \in C^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a simply connected region.[15] Suppose there exists a continuously differentiable function $\theta(x_1, x_2) \in C^1(\Omega)$ such that

$$\nabla \cdot (\theta\, F) = \frac{\partial(\theta f)}{\partial x_1} + \frac{\partial(\theta g)}{\partial x_2}$$

is either strictly positive or strictly negative throughout Ω. Then, the system admits no closed orbit entirely contained in Ω.

Proof. Assume, by contradiction, that there exists a closed orbit $\Gamma \subset \Omega$. Without loss of generality, suppose

$$\frac{\partial(\theta f)}{\partial x_1} + \frac{\partial(\theta g)}{\partial x_2} > 0 \quad \text{for all } (x_1, x_2) \in \Omega.$$

Then, applying Green's Theorem[16] to the vector field $(\theta f, \theta g)$ over the region $\mathcal{D}$ enclosed by Γ gives:

$$\iint_{\mathcal{D}} \left(\frac{\partial(\theta f)}{\partial x_1} + \frac{\partial(\theta g)}{\partial x_2} \right) dx_1\, dx_2 = \oint_{\Gamma} -\theta g\, dx_1 + \theta f\, dx_2.$$

[15] A region $\Omega \subset \mathbb{R}^2$ is said to be "simply connected" if it is connected and every simple closed curve in Ω encloses a region entirely contained in Ω; in other words, Ω has no holes. This condition ensures the applicability of Green's Theorem.

[16] Green's Theorem states that, for a positively oriented, piecewise smooth, simple closed curve Γ enclosing a region $\mathcal{D} \subset \mathbb{R}^2$, and for continuously differentiable functions P and Q, we have:

$$\oint_{\Gamma} P\, dx_1 + Q\, dx_2 = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1\, dx_2.$$

In our case, we rewrite the divergence as a line integral by taking $P = -\theta g$ and $Q = \theta f$.

Since the integrand on the left-hand side is strictly positive and $\mathcal{D}$ has nonzero area, the double integral is strictly positive. However, the right-hand side vanishes along Γ, because the vector field (f, g) is tangent to the trajectory and thus orthogonal to the outward normal:

$$\begin{aligned}\oint_\Gamma -\theta g\,dx_1 + \theta f\,dx_2 &= \int_0^T \left(-\theta g\frac{dx_1}{dt} + \theta f\frac{dx_2}{dt}\right) dt \\ &= \int_0^T (-\theta g f + \theta f g)\,dt \\ &= \int_0^T \theta \underbrace{(-g f + f g)}_{=(f,g)\cdot\mathbf{n}}\,dt \\ &= 0.\end{aligned}$$

This yields a contradiction. Therefore, no closed orbit can exist in Ω. □

Remark 5. Let Γ be a closed curve parametrized by $\mathbf{z}(t) = (z_1(t), z_2(t))$ for $t \in [a, b]$, with $\mathbf{z}(a) = \mathbf{z}(b)$. The line integral of a vector field (P, Q) along Γ is defined as

$$\oint_\Gamma P\,dz_1 + Q\,dz_2 = \int_a^b \left[P(z_1(t), z_2(t))\frac{dz_1}{dt} + Q(z_1(t), z_2(t))\frac{dz_2}{dt}\right] dt.$$

It measures the circulation or work done by the vector field (P, Q) along the closed path Γ.

Remark 6. Bendixson's Theorem [17] corresponds to Dulac's when $\theta = 1$.

[17] In 1933, the French mathematician Henri Dulac established a generalization of Bendixson's criterion.

Example

Example 4.4.2. Consider the system

$$x_1' = x_2,$$
$$x_2' = -x_1 - x_2 + x_1^2 + x_2^2.$$

Here $f(x_1, x_2) = x_2$ and $g(x_1, x_2) = -x_1 - x_2 + x_1^2 + x_2^2$.

To apply Bendixson–Dulac's criterion, we introduce the Dulac function

$$\theta(x_1, x_2) = e^{\alpha x_1}, \qquad \alpha \in \mathbb{R}.$$

Computing the divergence,

$$\frac{\partial}{\partial x_1}\big(\theta f\big) + \frac{\partial}{\partial x_2}\big(\theta g\big) = \frac{\partial}{\partial x_1}(e^{\alpha x_1} x_2) + \frac{\partial}{\partial x_2}\big(e^{\alpha x_1}(-x_1 - x_2 + x_1^2 + x_2^2)\big)$$
$$= e^{\alpha x_1}\big(\alpha x_2\big) + e^{\alpha x_1}\big(-1 + 2x_2\big).$$

Collecting terms gives

$$\frac{\partial}{\partial x_1}\big(\theta f\big) + \frac{\partial}{\partial x_2}\big(\theta g\big) = e^{\alpha x_1}\,\big(\alpha x_2 - 1 + 2x_2\big).$$

Choosing $\alpha = -2$, we obtain

$$\frac{\partial}{\partial x_1}\big(\theta f\big) + \frac{\partial}{\partial x_2}\big(\theta g\big) = -\,e^{-2x_1} < 0, \qquad \forall\,(x_1, x_2) \in \mathbb{R}^2.$$

Thus, by Bendixson–Dulac's criterion, the system does not admit closed orbits anywhere in the plane.

At this stage, we have introduced tools that, in certain cases, allow us to rule out the existence of closed orbits in a dynamical system. Our objective now, however, is not to exclude such orbits but rather to identify the conditions under which they do exist.

From this point forward, we shift the focus away from the stability of fixed points (equilibrium points, which were extensively analyzed in the preceding sections). Instead, we turn to a different type of attractor toward which trajectories may converge: the limit cycle.

Informally, a limit cycle is a closed, isolated trajectory in the phase space. Unlike a center, where all nearby trajectories are also closed, a limit cycle is surrounded by trajectories that are not closed: some approach the cycle while others diverge away from it. This behavior arises naturally in diverse disciplines, including physics (oscillatory systems), biology (population dynamics), and economics (models of business cycles and self-sustained oscillations).

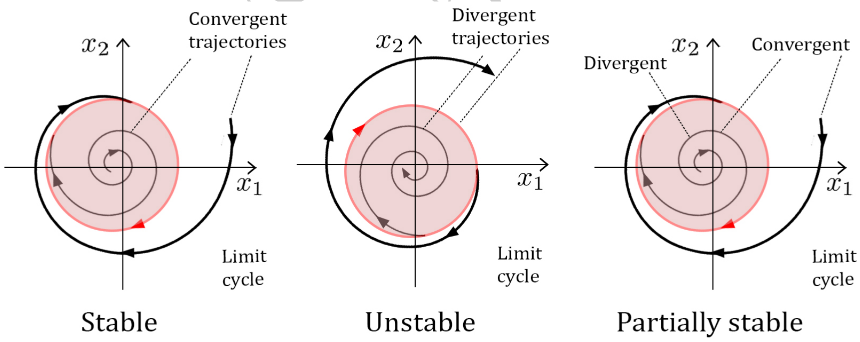

Figure 4.32 The three classical types of limit cycles

As shown in Figure 4.32, classically, three main types of limit cycles are distinguished:

- *Stable limit cycles*, where nearby trajectories spiral toward the cycle;
- *Unstable limit cycles*, where nearby trajectories spiral away from it;
- *Semi-stable limit cycles*, where trajectories approach the cycle on one side but diverge on the other.

To proceed rigorously, we next introduce a set of definitions that will allow us to formalize the concept of limit cycle and to state the central result of this section.

Definition 4.4.4. Let $I \subset \mathbb{R}$ be an open interval and let $\boldsymbol{\gamma} : I \to \mathbb{R}^n$ be a continuous function. The set

$$\Gamma = \{\boldsymbol{\gamma}(t) \in \mathbb{R}^n : t \in I\}$$

is called a *curve* in $\mathbb{R}^n$.

Definition 4.4.5. Let $F : \mathcal{U} \to \mathbb{R}^n$ be a continuously differentiable vector field on an open set $\mathcal{U} \subset \mathbb{R}^n$. The *flow* associated with the dynamical system

$$\mathbf{x}' = F(\mathbf{x})$$

is the map $\boldsymbol{\phi} : \Omega \subset \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ defined by

$$\boldsymbol{\phi}_t(\mathbf{x}_0) = \boldsymbol{\phi}(t, \mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}_0),$$

where $\mathbf{x}(t, \mathbf{x}_0)$ denotes the unique solution to the initial value problem with $\mathbf{x}(0) = \mathbf{x}_0$.

The domain Ω consists of all pairs $(t, \mathbf{x}_0)$ for which the solution exists. A *solution* refers to the trajectory corresponding to a fixed initial point $\mathbf{x}_0$, while the *flow* describes simultaneously the evolution of all such trajectories for different initial conditions.

As explained above, fixing an initial point $\mathbf{x}_0$ and letting time vary produces the trajectory

$$t \;\mapsto\; \mathbf{x}(t) \;=\; \boldsymbol{\phi}(t, \mathbf{x}_0).$$

By contrast, fixing a time t_0 and varying the initial condition defines the *flow map at time* t_0:

$$\mathbf{x}_0 \;\mapsto\; \boldsymbol{\phi}(t_0, \mathbf{x}_0),$$

which advances every point of the phase space forward by the same duration t_0. It is important to emphasize that $\boldsymbol{\phi}(t,\mathbf{x}_0)$ depends explicitly on two arguments—time t and initial condition $\mathbf{x}_0$—and is not a single-variable function.

Concept	**Input(s)**	**Description**
Trajectory	t (with $\mathbf{x}_0$ fixed)	The path followed by a single initial state $\mathbf{x}_0$ as time evolves: $t \mapsto \mathbf{x}(t) = \boldsymbol{\phi}(t, \mathbf{x}_0)$.
Flow map at time t	$(t, \mathbf{x}_0)$	The mapping that takes each initial state $\mathbf{x}_0$ to its position after t units of time: $\mathbf{x}_0 \mapsto \boldsymbol{\phi}(t, \mathbf{x}_0)$.

Table 4.1 Comparison between a trajectory and the flow map at time t.

Example

Example 4.4.3. Consider the planar system

$$\mathbf{x}' = F(\mathbf{x}) = (-x_2, x_1),$$

which generates uniform circular motion around the origin. For the initial condition $\mathbf{x}_0 = (1{,}0)$, the corresponding trajectory is

$$\mathbf{x}(t) = (\cos t,\, \sin t).$$

More generally, the flow map is given by

$$\boldsymbol{\phi}(t, (x_{10}, x_{20})) = \big(x_{10}\cos t - x_{20}\sin t, x_{10}\sin t + x_{20}\cos t\big).$$

In particular, for $t = \pi/2$ we obtain

$$\boldsymbol{\phi}\big(\tfrac{\pi}{2}, (1, 0)\big) = (0, 1),$$

showing that the flow sends the point $(1, 0)$ to $(0, 1)$ after a quarter turn.

In fact, the map $\boldsymbol{\phi}(t,\cdot)$ can be interpreted as the rotation operator on the plane by angle t.

◊◊◊

Definition 4.4.6. (Region). A region in $\mathbb{R}^n$ refers to a non-empty, open, and connected subset of $\mathbb{R}^n$.

Definition 4.4.7. (Attracting set). Let $\boldsymbol{\phi}$ denote the flow of the dynamical system $\mathbf{x}' = F(\mathbf{x})$. A closed invariant set $X \subset \Omega$ is called an *attracting set* if there exists a neighborhood $\mathcal{N} \subset \Omega$ of X such that:

1. (*Forward invariance*) Every trajectory starting in $\mathcal{N}$ remains in $\mathcal{N}$ for all future times:

$$\boldsymbol{\phi}_t(\mathbf{x}) \in \mathcal{N} \quad \text{for all } t \geq 0,\ \mathbf{x} \in \mathcal{N}.$$

2. (*Asymptotic attraction*) Every trajectory starting in $\mathcal{N}$ approaches X asymptotically:

$$\lim_{t\to\infty} \operatorname{dist}\big(\boldsymbol{\phi}_t(\mathbf{x}), X\big) = 0, \quad \text{for all } \mathbf{x} \in \mathcal{N},$$

where $\operatorname{dist}(\mathbf{y}, X) := \inf_{\mathbf{z}\in X} \|\mathbf{y} - \mathbf{z}\|$ denotes the Euclidean distance from $\mathbf{y}$ to X.

Example

Example 4.4.4. Consider the one-dimensional system

$$x' = -x.$$

The flow is $\phi(t, x_0) = x_0 e^{-t}$, which converges to 0 as $t \to \infty$ for every initial condition $x_0 \in \mathbb{R}$. Here the attracting set is simply $X = \{0\}$, since all trajectories approach the origin.

Economic interpretation: this corresponds to a situation where an economy subject to shocks or deviations always converges back to a steady state (the equilibrium at 0). The equilibrium acts as an attracting set for the dynamics of the system.

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Attracting sets are often detected via the notion of a *trapping region*.

Definition 4.4.8. (Trapping region). A closed and connected region $\mathcal{R}$ is called a *trapping region* if

$$\boldsymbol{\phi}_t(\mathcal{R}) \subset \mathcal{R} \quad \text{for all } t \geq 0.$$

Equivalently, the vector field F points strictly inward along the boundary of $\mathcal{R}$.

Remark 7. Given a trapping region $\mathcal{R}$, the associated attracting set is

$$X \;=\; \bigcup_{t\geq 0} \boldsymbol{\phi}_t(\mathcal{R}). \tag{4.4.11}$$

See Exercise 4.4.3.

Definition 4.4.9. (Omega-limit set). Let $\boldsymbol{\phi} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow of the system

$$\mathbf{x}' = F(\mathbf{x}),$$

and fix an initial condition $\mathbf{x}_0 \in \mathbb{R}^n$. A point $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is called an *omega-limit point* of $\mathbf{x}_0$ if there exists a sequence of times $(t_n)_{n\in\mathbb{N}}$ with $t_n \to +\infty$ such that

$$\boldsymbol{\phi}(t_n, \mathbf{x}_0) \;\to\; \tilde{\mathbf{x}} \quad \text{as } n \to \infty.$$

The collection of all omega-limit points of $\mathbf{x}_0$ is called the *omega-limit set* of $\mathbf{x}_0$, denoted

$$\omega(\mathbf{x}_0) \;=\; \{\, \tilde{\mathbf{x}} : \exists\, t_n \to +\infty,\, \boldsymbol{\phi}(t_n,\mathbf{x}_0) \to \tilde{\mathbf{x}}\}.$$

Equivalently,

$$\omega(\mathbf{x}_0) \;=\; \bigcap_{t\geq 0} \overline{\{\boldsymbol{\phi}(s, \mathbf{x}_0) : s \geq t\}},$$

where the overline denotes closure. In the planar case ($n = 2$), omega-limit sets are restricted to three possibilities:

- an equilibrium point,
- a periodic orbit (limit cycle), or
- a finite union of trajectories connecting saddle points (saddle connections).

Example

Example 4.4.5. Consider the logistic equation

$$x' = x(1 - x).$$

For any initial condition $x_0 > 0$, the solution stays in the interval $(0,\infty)$. If $0 < x_0 < 1$, then $x(t)$ remains in $(0, 1)$ for all $t \geq 0$, so $(0, 1)$ is a trapping region. The attracting set is $X = \{1\}$, since every trajectory converges to the equilibrium $x = 1$. In terms of omega-limit sets, for any $x_0 \in (0,1)$ we have $\omega(x_0) = \{1\}$.

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Remark 8. A stable equilibrium point $\mathbf{x}^*$ is itself an omega–limit set: for every initial condition $\mathbf{x}_0$ in its basin of attraction,

$$\lim_{t\to+\infty} \boldsymbol{\phi}(t,\mathbf{x}_0) = \mathbf{x}^*,$$

so that $\omega(\mathbf{x}_0) = \{\mathbf{x}^*\}$.

By contrast, if $\omega(\mathbf{x}_0)$ contains a periodic orbit of period $T > 0$, then different subsequences $t_n \to +\infty$ (taken modulo T) may converge to different points on the same cycle. In this case the omega–limit set is not a singleton but the entire closed trajectory of the limit cycle.

We now turn to the formal definition of a limit cycle and to the precise statement of the Poincaré–Bendixson Theorem.

Definition 4.4.10. (Limit cycle). A *limit cycle* is a closed and isolated orbit $\Gamma \subset \Omega \subset \mathbb{R}^2$ such that there exists at least one initial condition $\mathbf{x}_0$ whose omega–limit set is exactly Γ.

Equivalently, there exists a neighborhood $\mathcal{N}$ of Γ and a point $\mathbf{x} \in \mathcal{N}$ such that

$$\lim_{t\to\infty} \|\boldsymbol{\phi}_t(\mathbf{x}) - \boldsymbol{\gamma}(t)\| = 0,$$

where $\boldsymbol{\gamma}(t)$ parametrizes Γ.

Example

Example 4.4.6. Limit cycles arise naturally in nonlinear systems, but they are impossible in certain settings. First, if a system admits a globally attracting equilibrium, then the omega–limit set of every trajectory is that equilibrium, leaving no room for an isolated closed orbit. Second, no *linear system* can have a limit cycle. Consider

$$\mathbf{x}' = A\,\mathbf{x}, \quad A \in \mathcal{M}_{n\times n}.$$

If $\mathbf{x}(t)$ is a nontrivial periodic solution, then for any nonzero scalar α the rescaled trajectory $\alpha\,\mathbf{x}(t)$ is also periodic and can be made arbitrarily close to the original orbit. Thus, no periodic trajectory is isolated, which rules out limit cycles.

Figure 4.33 illustrates this property: trajectories corresponding to different scalings overlap, showing the absence of isolated closed orbits in linear systems.

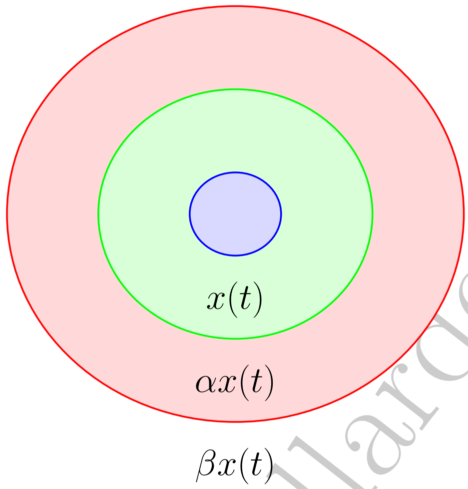

Figure 4.33 Absence of isolated closed orbits in a linear system: scaling preserves periodicity

◊◊◊

We are now ready to state the Poincaré–Bendixson Theorem, which provides sufficient conditions for the existence of a periodic orbit in a simply connected region of the plane.

Theorem 19. (Poincaré–Bendixson Theorem). Let $\Omega \subset \mathbb{R}^2$ be a simply connected open region, and consider

$$\mathbf{x}' = F(\mathbf{x}) = (f(x_1,x_2), g(x_1,x_2)), \qquad F \in C^1(\Omega).$$

If for some initial condition $\mathbf{x}_0 \in \Omega$ the omega–limit set $\omega(\mathbf{x}_0)$ is nonempty, compact, and contains no equilibrium points, then $\omega(\mathbf{x}_0)$ is a periodic orbit (i.e., a closed trajectory).

Remark 9. Intuitively, Theorem 19 ensures the following: if a trajectory starting at $\mathbf{x}_0$ remains forever inside Ω, that is,

$$\forall\, t \geq 0, \quad \boldsymbol{\phi}_t(\mathbf{x}_0) \in \Omega \quad \big(\text{since } \omega(\mathbf{x}_0) \subset \Omega\big),$$

then the trajectory is either a closed orbit itself, or it spirals toward a closed path Γ, which by Definition 4.4.10 is a limit cycle.

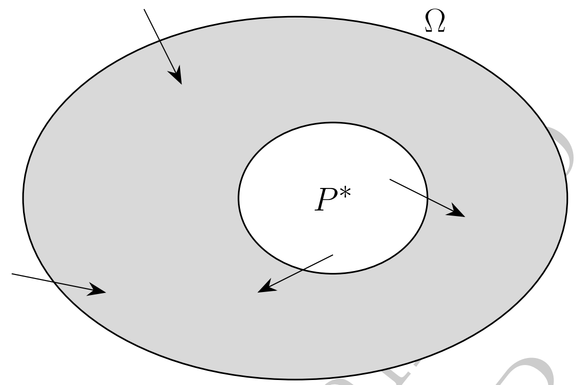

Figure 4.34 A trapping region: trajectories cannot escape Ω, and the equilibrium is excluded

Remark 10. The Poincaré–Bendixson Theorem is one of the cornerstones of planar dynamics. It provides a powerful qualitative tool, but it applies only in two dimensions and may be difficult to use in practice. In economics, this restriction is rarely problematic, since many dynamic models of interest are two-dimensional. For detailed proofs, see Perko (2013) or Viana and Espinar (2021).

Practical procedure for applying the theorem. In applications, one usually proceeds as follows:

- Locate the equilibrium point(s) P^* and study their stability using linearization (see Section 4.2).

- If P^* is unstable, select a bounded region Ω that isolates it. Geometrically, this can be an annulus whose inner boundary encloses P^*.

- Construct the outer boundary of Ω so that the vector field F points strictly inward.
- On the inner boundary (near the unstable equilibrium), ensure that the vector field points into Ω, i.e., away from P^*.
- Then Ω is a trapping region. If $F \in C^1(\Omega)$, the hypotheses of the theorem are satisfied, guaranteeing the existence of a closed orbit in Ω.

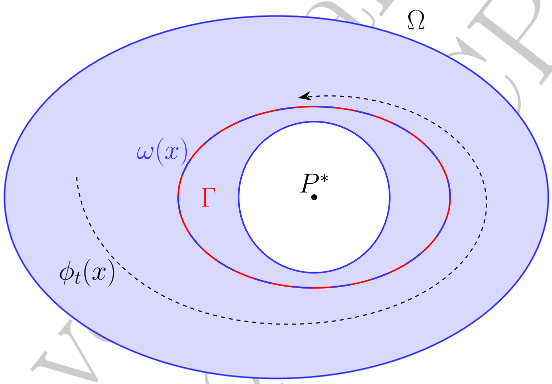

Figure 4.35 Construction of a trapping region for the application of the Poincaré–Bendixson Theorem

Examples

Example 4.4.7. *(Kaldor's nonlinear cyclic model).* When Keynes published *The General Theory of Employment, Interest and Money* (1936), several economists (e.g., Samuelson, Hicks) attempted to formalize business cycles. In 1940, Nicolas Kaldor proposed a nonlinear model generating self-sustained cycles.[18] Kaldor's ideas are related to Kalecki's and build on Keynes's framework. Investment and saving are modeled as C^2 functions of output Y and capital K:

$$I = I(Y,K), \qquad S = S(Y,K),$$

with a strictly decreasing marginal efficiency of capital. We consider a closed economy, zero government spending, and a positive depreciation rate $\delta > 0$ (following Lorenz (1993)). The exposition follows Gandolfo (2009).

Hypotheses and their economic interpretation.

a) $I_Y > 0$ (positive marginal propensity to invest).
   Meaning: Higher income raises capacity utilisation, profits, and expected sales; via the accelerator and cash–flow channels, firms invest more when Y rises.

b) There exists a "normal" income range $[Y_1, Y_2]$ with $I_{YY} > 0$ for $Y \in [Y_1, Y_2]$ (outside this range, I_Y declines).
   Meaning: Within typical utilisation rates, investment becomes more sensitive to income (backlogs, confidence, easing financial constraints). In deep recessions ($Y < Y_1$), credit constraints and weak prospects reduce I_Y; in booms ($Y > Y_2$), bottlenecks and rising costs dampen I_Y.

[18]Cyclical behavior later evolved through Real Business Cycle (RBC) theory (e.g., Romer (2019), Galí (2015)), which we do not treat here because its methodology is discrete dynamic optimization.

c) $I_K < 0$ (diminishing marginal efficiency of capital).
Meaning: A larger installed capital stock lowers the marginal payoff of additional capital (saturation/capacity effects). Firms need less expansion investment when K is already high.

d) $S_Y > 0$ and $S_K > 0$ (saving increases with income and capital).
Meaning: A higher income raises saving out of current income; a larger capital stock generates property income (profits, interest), lifting aggregate saving.

e) As $Y > Y_2$, the saving rate rises; as $Y < Y_1$, saving collapses (limited ability to smooth consumption).[19]
Meaning: At high incomes, households choose higher saving shares (precautionary/wealth accumulation motives). Near subsistence or under tight borrowing constraints, households dissave or save little.

f) Dynamics:

$$\begin{aligned} Y' &= \alpha\,[I(Y,K) - S(Y,K)], \qquad \alpha > 0, \\ K' &= I(Y,K) - \delta K. \end{aligned} \tag{4.4.12}$$

Meaning: Goods-market adjustment: output changes with the investment–saving gap; α is the speed of adjustment (inventory/multiplier dynamics in continuous time). Capital accumulates with net investment $I - \delta K$. δK is capital depreciation.

[19] See, e.g., Friedman (1957), Hall (1978), Mehra and Prescott (1985).

g) Instability in the normal range:

$$S_Y < I_Y \quad \text{for } Y \in [Y_1, Y_2], \tag{4.4.13}$$

and the opposite relation holds outside the normal range. Moreover,

$$\alpha\,(I_Y - S_Y) + I_K - \delta > 0. \tag{4.4.14}$$

Meaning: In the normal range, investment reacts more to income than saving does, so a rise in Y produces excess demand and further increases in Y (self-reinforcing "accelerator" feedback). The broader condition ensures the feedback (via I_Y) minus dilution (δ) and diminishing returns ($I_K < 0$) still yields a locally unstable steady state, opening the door to cycles. This last is more a technical requirement.

Assumptions a)-g) deliver the shapes of $I(\cdot)$ and $S(\cdot)$ (Figure 4.36).

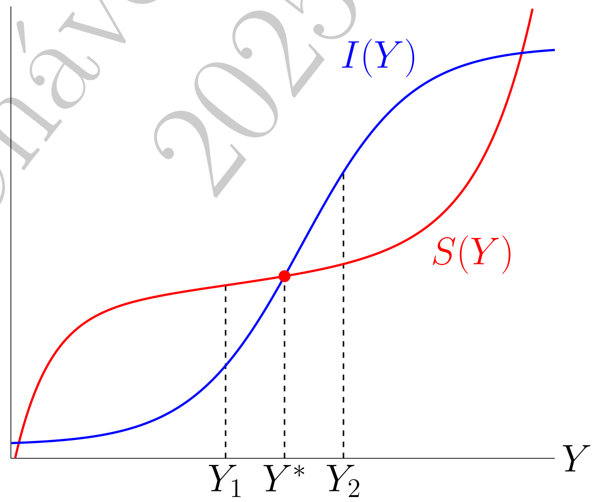

Figure 4.36 Investment and saving as functions of output Y

Equilibrium. At a nontrivial steady state (Y^*,K^*), (4.4.12) implies

$$I(Y^*,K^*) = \delta K^*, \qquad S(Y^*,K^*) = \delta K^*.$$

Since $S(Y,K^*)$ is strictly increasing in Y, the equation $S(Y,K^*) = \delta K^*$ has a unique solution Y^*. With the standard shape (slowly increasing/concave for $Y < Y_1$ and rapidly increasing/convex for $Y > Y_2$), this unique Y^* lies in $[Y_1,Y_2]$.

Local stability. The Jacobian at (Y^*,K^*) is

$$J(Y^*,K^*) = \begin{bmatrix} \alpha\,(I_Y - S_Y) & \alpha\,(I_K - S_K) \\ I_Y & I_K - \delta \end{bmatrix},$$

with characteristic polynomial $\lambda^2 + c_1\lambda + c_2 = 0$, where

$$c_1 = -(\alpha\,(I_Y - S_Y) + I_K - \delta), \qquad c_2 = \det J.$$

The determinant is

$$\begin{aligned} \det J &= \alpha\Big[(I_Y - S_Y)(I_K - \delta) - I_Y(I_K - S_K)\Big] \\ &= \alpha\Big(I_Y S_K - S_Y I_K + \delta(S_Y - I_Y)\Big). \end{aligned}$$

Under $I_Y > 0$, $S_Y > 0$, $I_K < 0$, $S_K > 0$, we have $I_Y S_K - S_Y I_K > 0$. In the normal range, (4.4.13) gives $S_Y - I_Y < 0$, hence $\delta(S_Y - I_Y) < 0$. In many reasonable parametrizations the positive term dominates, yielding $\det J > 0$.[20] By (4.4.14), $\alpha\,(I_Y - S_Y) + I_K - \delta > 0$, hence $c_1 < 0$ and $\operatorname{Tr} J = -c_1 > 0$. With $\det J > 0$ and $\operatorname{Tr} J > 0$, the equilibrium is unstable (eigenvalues with positive real parts).

[20] To ensure $\det J > 0$ without extra functional assumptions, impose the sufficient bound $I_Y S_K - S_Y I_K > \delta\,(I_Y - S_Y)$.

Vector field and isoclines. From $Y' = 0$ we have $I(Y, K) = S(Y, K)$. Total differentiation w.r.t. Y yields

$$I_K \frac{dK}{dY} + I_Y = S_K \frac{dK}{dY} + S_Y \quad \Rightarrow \quad \left.\frac{dK}{dY}\right|_{Y'=0} = \frac{S_Y - I_Y}{I_K - S_K}.$$

Since $I_K < 0$ and $S_K > 0$, the denominator is negative. In the normal range, (4.4.13) gives $S_Y - I_Y < 0$, so

$$\left.\frac{dK}{dY}\right|_{Y'=0} > 0 \quad \text{for } Y \in [Y_1,Y_2].$$

Outside the normal range the sign flip since $S_Y > I_Y$.

For $K' = 0$ (i.e., $I(Y,K) = \delta K$), total differentiation gives

$$I_K \frac{dK}{dY} + I_Y = \delta \frac{dK}{dY} \quad \Rightarrow \quad \left.\frac{dK}{dY}\right|_{K'=0} = -\frac{I_Y}{I_K - \delta}.$$

Since $I_Y > 0$ and $I_K - \delta < 0$, we obtain $dK/dY|_{K'=0} > 0$; in the normal range I_Y is larger, so this slope is steeper. These signs yield the isoclines in Figure 4.37.

Additional hypotheses Chang and Smyth (1971). Assume there exists $\tilde{K} > 0$ with $I(0,\tilde{K}) = 0$ and $\tilde{Y} > 0$ with $I(\tilde{Y},0) = S(\tilde{Y},0)$. Together with the sign changes of Y',K' only when crossing their isoclines, this pins down the vector field in Figure 4.37.

Kaldor via Poincaré–Bendixson. Let

$$\Omega = \{(Y,K) \in \mathbb{R}^2_{++} : \ 0 \le Y \le \bar{Y},\ 0 \le K \le \bar{K}\} \setminus \mathcal{B}(P^*,\varepsilon),$$

with small $\varepsilon > 0$ and $\partial\mathcal{B}$ oriented toward P^*. Then:

- P^* is unstable (see above).
- By construction, $P^* \notin \Omega$ and the field points inward on $\partial\Omega$; hence Ω is a trapping region.

- $I, S \in C^1(\Omega)$ by assumption.

By the Poincaré–Bendixson Theorem, a limit cycle exists in Ω.

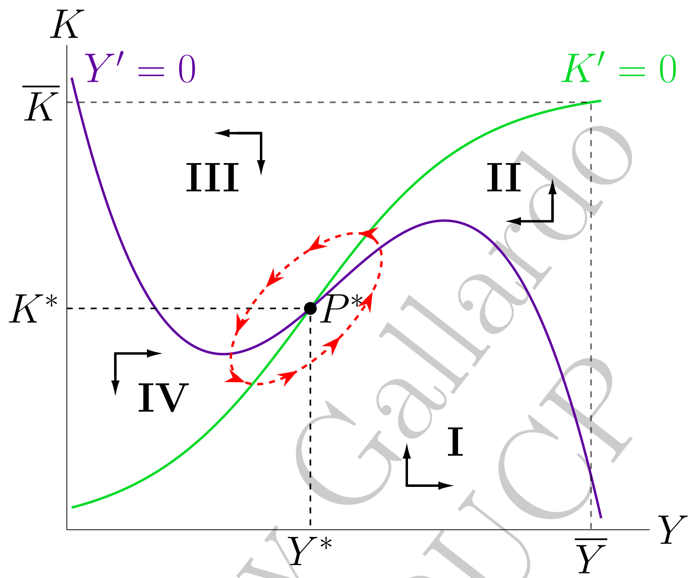

Figure 4.37 Vector field and isoclines for Kaldor's model

Economic analysis. In Figure 4.37 (positive quadrant), four "isozones" emerge:

a) *I:* K and Y low $\Rightarrow$ high marginal productivity of capital; $I > S$ $(Y' > 0)$. Trajectories move to higher levels of K and Y (boom).

b) *II:* after crossing $Y' = 0$, $Y' < 0$ (high saving for $Y > Y_2$), but still $K' > 0$ (adjustment lags). Leftward motion while K keeps rising.

c) *III:* after crossing $K' = 0$, $K' < 0$ (net disinvestment) and output contracts; recession.

d) *IV:* crossing $Y' = 0$ again, saving is exhausted and $I - S > 0$ (credit-financed). The economy returns to I; low rate K/Y restores high marginal productivity, restarting the cycle.

Conclusions and critique. Kaldor's nonlinear model generates self-sustained (Y, K)-cycles rather than convergence to (Y^*, K^*). With $\delta > 0$, capital depreciates at rate δK and $K' = I - \delta K$ can be negative even with $I > 0$. The mechanism captures boom–bust phases, but it does not explain the long-run growth trend. Subsequent chapters study microfounded models (intertemporal optimization and equilibrium) that better match data outside extreme episodes.

Example 4.4.8. *(Cross–dual adjustment process).* This example generalizes Walras's price adjustment under excess demand. Let p denote the price and y the (actual) output. Prices respond to excess demand $\xi = D(p) - y$, and output adjusts toward desired supply $S(p)$. The general form is

$$p' = \psi\big(D(p) - y\big), \qquad y' = \varphi\big(S(p) - y\big),$$

where ψ and φ are strictly increasing functions. In the linear-response specification (a standard simplification),

$$\begin{aligned} p' &= \alpha\,\big(D(p) - y\big), \\ y' &= \beta\,\big(S(p) - y\big), \end{aligned} \tag{4.4.15}$$

with $\alpha, \beta > 0$. The state is $(p, y) \in \mathbb{R}^2_+$ and the vector field is $F(p, y) = (\alpha(D(p) - y), \beta(S(p) - y))$.

Hypotheses and their economic interpretation.

a) *Local nonstandard demand slope (aggregation effect).* There exists a neighborhood $\mathcal{N}(p^*)$ of the equilibrium price p^* such that

$$D_p(p) > \frac{\beta}{\alpha} > 0 \quad \text{for } p \in \mathcal{N}(p^*).$$

Meaning: Although individual demand typically has $D_p < 0$, aggregation can generate locally upward-sloping aggregate demand near p^* (Mas-Colell (1986); see also Lorenz (1993)).

b) *Supply at zero price.* $S(0) = 0$. *Meaning:* with zero price, desired output is nil.

c) *Unique equilibrium with supply steeper than demand at p^*.* There is a unique (p^*, y^*) such that $D(p^*) = S(p^*) = y^*$ and

$$S_p(p^*) > D_p(p^*).$$

Meaning: the supply curve crosses the demand curve once and is locally steeper at the intersection.

d) *Global monotonicities (away from $\mathcal{N}(p^*)$).* $D_p(p) < 0$ for $p \notin \mathcal{N}(p^*)$; $S_p(p) > 0$ for all $p \geq 0$. *Meaning:* demand is downward sloping except possibly near p^*; supply is upward sloping.

e) *Axis intercepts (outer bounds).* $D(0) = \bar{y} > 0$ and there exists $\bar{p} > 0$ with $D(\bar{p}) = 0$. *Meaning:* the demand curve hits the axes so we can choose a rectangle $[0, \bar{p}] \times [0, \bar{y}]$ containing dynamics of interest.

f) *First–quadrant invariance at the axes.* On $y = 0$, $y' = \beta S(p) \geq 0$ (since $S(0) = 0$ and S is increasing); on $p = 0$,

$p' = \alpha(D(0)-y) \geq 0$ for $y \leq \bar{y} = D(0)$. *Meaning:* trajectories touching the axes are pushed into the interior of the first quadrant.

Equilibrium analysis. At the unique steady state (p^*,y^*) we have $D(p^*) = S(p^*) = y^*$. The Jacobian of (4.4.15) at (p^*,y^*) is

$$J(p^*,y^*) = \begin{bmatrix} \alpha D_p(p^*) & -\alpha \\ \beta S_p(p^*) & -\beta \end{bmatrix}.$$

Hence

$$\det J = \alpha\beta\big(S_p(p^*) - D_p(p^*)\big) > 0 \quad \text{(by c)},$$

and

$$\operatorname{tr} J = \alpha D_p(p^*) - \beta > 0 \quad \text{(by a)}.$$

Therefore the equilibrium is *unstable*: both eigenvalues have positive real part, so trajectories are repelled from (p^*, y^*).

Vector field and isoclines. The nullclines are $y = D(p)$ (for $p' = 0$) and $y = S(p)$ (for $y' = 0$). The plane is partitioned into four regions with the following sign pattern:

- $y < D(p)$ and $y > S(p)$: $p' > 0$, $y' < 0$ (right–down).
- $y < D(p)$ and $y < S(p)$: $p' > 0$, $y' > 0$ (right–up).
- $y > D(p)$ and $y < S(p)$: $p' < 0$, $y' > 0$ (left–up).
- $y > D(p)$ and $y > S(p)$: $p' < 0$, $y' < 0$ (left–down).

Economically: when demand exceeds output ($y < D(p)$), price rises ($p' > 0$); when desired supply exceeds output ($y < S(p)$), output rises ($y' > 0$), etc.

A trapping region and Poincaré–Bendixson. Choose $\bar{p} > 0$ with $D(\bar{p}) = 0$ and let $\bar{y} \geq \max\{D(0), S(\bar{p})\}$. Consider $\Omega = \big([0,\bar{p}] \times [0,\bar{y}]\big) \setminus \mathcal{B}((p^*,y^*),r)$, where $r > 0$ is small. On the outer boundary:

$$\begin{aligned}
p = 0 :& \quad p' = \alpha(D(0) - y) \geq 0 \ \text{ for } y \in [0,\bar{y}],\\
p = \bar{p} :& \quad p' = \alpha(D(\bar{p}) - y) = -\alpha y \leq 0,\\
y = 0 :& \quad y' = \beta S(p) \geq 0 \ \text{ for } p \in [0,\bar{p}],\\
y = \bar{y} :& \quad y' = \beta(S(p) - \bar{y}) \leq 0 \ \text{ since } \bar{y} \geq S(\bar{p}) \geq S(p).
\end{aligned}$$

Thus F points inward along the outer boundary. On the inner boundary $\partial\mathcal{B}((p^*,y^*),r)$, instability implies there exists $r > 0$ such that

$$F(p,y) \cdot \frac{(p - p^*, y - y^*)}{\|(p - p^*, y - y^*)\|} \ > \ 0,$$

i.e., the field points *away* from the equilibrium and therefore *into* Ω across the inner boundary. Hence Ω is a trapping region containing no equilibrium. By the Poincaré–Bendixson Theorem, there exists a periodic orbit in Ω.

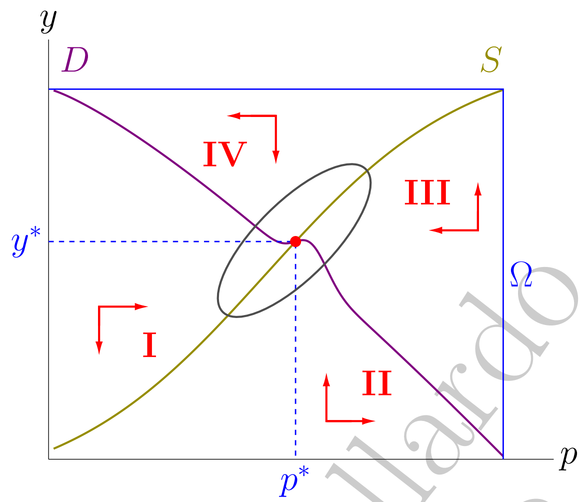

Figure 4.38 Cross–dual adjustment: vector field and nullclines $y = D(p)$ and $y = S(p)$

Conclusions. Under the aggregation-induced local demand slope $D_p(p^*) > \beta/\alpha$ and a steeper supply $S_p(p^*) > D_p(p^*)$, the unique equilibrium is unstable. With inward pointing flow on an outer rectangle and outward flow on a small circle around (p^*,y^*), a trapping region arises; the Poincaré–Bendixson Theorem then guarantees a limit cycle. The cycle captures persistent price–output oscillations generated by cross adjustments of prices to excess demand and output to desired supply.

◇◇◇

Before concluding this section, we state one final result, essential for justifying the existence of equilibria in the interior of the region bounded by a limit cycle.

Theorem 20. (Poincaré Theorem). Consider the planar system

$$x_1' = f(x_1, x_2),$$
$$x_2' = g(x_1, x_2).$$

Every closed orbit Γ necessarily surrounds at least one equilibrium point.

Proof. Let Γ be a closed orbit of the system. By uniqueness of solutions for C^1 vector fields, Γ is a simple closed curve. Define Ω the area surrounded by Γ. This is $\Gamma = \partial\Omega$. Then, since $\mathbf{x}^* \notin \Omega^\circ$, $(f)^2 + (g)^2 \neq 0$ in Ω. Define also θ, the angle between the tangent to Γ and the $X-$axis. Clearly,

$$\oint_\Gamma d\theta = 2\pi. \tag{4.4.16}$$

On the other hand,

$$\tan\theta = \frac{x_2}{x_1} = \frac{g}{f} \tag{4.4.17}$$

$$\sec^2\theta\, d\theta = \left(1 + \tan^2\theta\right) d\theta = \frac{f dg - g df}{f^2}.$$

Thus, by (4.4.17), we have that

$$\left(1 + \frac{g^2}{f^2}\right) d\theta = \frac{f dg - g df}{f^2} \Leftrightarrow d\theta = \frac{f dg + g df}{f^2 + g^2}.$$

Integrating over the curve Γ we obtain

$$\oint_\Gamma d\theta = \oint_\Gamma \frac{f dg + g df}{f^2 + g^2}.$$

Then, by Green's Theorem

$$\oint_\Gamma \frac{f dg + g df}{f^2 + g^2} = \iint_\Omega \frac{\partial}{\partial f}\left(\frac{f}{f^2 + g^2}\right) + \frac{\partial}{\partial g}\left(\frac{g}{f^2 + g^2}\right)\, dx_1 dx_2.$$

Nonetheless

$$\frac{\partial}{\partial f}\left(\frac{f}{f^2+g^2}\right)+\frac{\partial}{\partial g}\left(\frac{g}{f^2+g^2}\right)=0. \tag{4.4.18}$$

Thus, from (4.4.16) and (4.4.18),

$$\oint_\Gamma d\theta = 2\pi = 0,$$

which is a contradiction. Hence, Γ surrounds at least one equilibrium. □

Theorem 20 implies that any closed trajectory in a planar dynamical system encircles at least one equilibrium. Equivalently, if $\Omega \subset \mathbb{R}^2$ contains no equilibria, then Ω cannot contain a closed trajectory.

Remark 11. A stronger statement holds: the *index* of a periodic orbit is +1, so the sum of the indices of all equilibria inside the closed orbit equals 1. In particular, the equilibria of both the Kaldor model and the cross–dual adjustment model lie inside the region bounded by their limit cycle Γ.

PROBLEM SET

Exercise 4.4.1. Prove that the planar nonlinear system

$$x_1' = \frac{x_1^2 - x_2^2}{(x_1^2+x_2^2)^2}, \tag{4.4.19}$$

$$x_2' = \frac{2\,x_1\,x_2}{(x_1^2+x_2^2)^2} \tag{4.4.20}$$

does not admit any closed orbits in

$$\Omega \;=\; \mathbb{R}^2 \setminus \{(0,0)\}.$$

Exercise 4.4.2. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathbf{x}_0 \in \Omega$ be such that for every $\mathbf{x} \in \Omega$ the line segment

$$\{(1-\theta)\,\mathbf{x}_0 + \theta\,\mathbf{x} \mid \theta \in [0,1]\} \subset \Omega.$$

Suppose $F\colon \Omega \to \mathbb{R}^n$ is a C^1 vector field whose Jacobian matrix $J_F(\mathbf{x})$ is symmetric for all $\mathbf{x} \in \Omega$. Define

$$\varphi(\mathbf{x}) = \int_0^1 F\big((1-t)\,\mathbf{x}_0 + t\,\mathbf{x}\big) \cdot (\mathbf{x} - \mathbf{x}_0)\,dt.$$

Prove that φ is a scalar potential for F, i.e. that

$$\nabla\varphi(\mathbf{x}) = F(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega.$$

Exercise 4.4.3. Prove the assertion done in Remark 4.4.
Hint: if $\mathbf{x} \in X$, then $\operatorname{dis}(\mathbf{x}, X) = 0$.

Exercise 4.4.4. Consider the following predator-prey model

$$\begin{aligned} x_1' &= \left(4 - x_1 - \frac{2x_2}{1+x_1}\right) \\ x_2' &= x_2(x_1 - 1). \end{aligned}$$

a) Argue why this system indeed represents the predator-prey model.

b) Assume that all solutions are bounded. Prove that this system has a limit cycle in $\mathbb{R}^2_+$.

Exercise 4.4.5. (Kaldor model revisited). With reference to Kaldor's model, explain why the saving and investment functions are represented graphically as in Figure 4.36.

Exercise 4.4.6. (Classical Cross–Dual Adjustment Process revisited). With respect to the cross–dual adjustment process model presented in Example 4.4.8, provide two examples of goods for which a price increase leads to a higher quantity demanded. For each case, explain how the substitution effect and the income effect must operate to generate this counter-intuitive outcome.